CD algebras

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CD algebras

CD algebras are anticommutative algebras such that $[R_a, R_b]$ is a derivation for any two elements a, b of the algebra.

Such commutative algebras = Lie triple algebras
(Bertram, Dzhumadil'daev, Jordan–Rühaak, Osborn, Sidorov, ..., 1969–2018)

What about *anti*commutative ones?

Known classes of anticommutative algebras

$$\begin{array}{ccc} & & \text{Malcev} \subset \text{Binary Lie} \\ & \rotatebox{45}{$\subset$} & \\ \text{Lie} & & \\ & \rotatebox{-45}{$\subset$} & \\ & & \text{Sagle} \end{array}$$

binary Lie: $J([x,y],x,y)=0$
Malcev: $J(x,y,[x,z])=[J(x,y,z),x]$
Sagle: $[J(x,y,z),t]=J(t,z,[x,y])+J(t,y,[z,x])+J(t,x,[y,z])$

How CD fits into the picture?

$$\begin{array}{ccccccc} & & & & & & \text{Binary Lie} \\ & & & & & \rotatebox{45}{$\subset$} & \\ \text{Lie} & \subset & \text{Malcev} \cap \text{Sagle} & \subset & \text{CD} & & \\ & & & & & \rotatebox{-45}{$\subset$} & \\ & & & & & & \text{Almost Lie} \end{array}$$

$$\text{Binary Lie} \cap \text{Almost Lie} = \text{CD}$$

Almost Lie: $[J(x,y,z),t] = 0$

CD algebras are "central extensions" of Lie algebras

Corollary 1 (to inclusion at the previous slide)

For any CD algebra A, $A/Z(A)$ is a Lie algebra.

Corollary 2

Any simple CD algebra is a Lie algebra.

An old open question

Whether any simple binary Lie algebra is Malcev?

Yes in the class of finite-dimensional algebras over a field of characteristic 0 (Grishkov 1980).
Yes in the class of CD algebras.

Cohomology? Variant 1

Central extensions + derivations + deformations

$$0 \to LZ(M) \oplus (A \otimes M) \xrightarrow{\mathrm{d}^0} C^1(A, M) \xrightarrow{\mathrm{d}^1} C^2(A, M) \xrightarrow{\mathrm{d}^2} C^4(A, M)$$

$LZ(M) = \{m \in M \,|\, xy \bullet m - x \bullet (y \bullet m) + y \bullet (x \bullet m) = 0 \forall x, y \in M\}$, the "Lie center"

$C^n(A, M)$ - skew-symmetric multilinear maps

$\mathrm{d}^0(m)(b) = b \bullet m$
$\mathrm{d}^0(a \otimes m)(b) = a \bullet (b \bullet m) - b \bullet (a \bullet m)$
$\mathrm{d}^1(\varphi)(x, y) = \varphi(xy) - x \bullet \varphi(y) + y \bullet \varphi(x)$
$\mathrm{d}^2(\varphi)(x, y, a, b) =$
$\varphi((xy)a, b) - \varphi((xy)b, a) - \varphi((xa)b, y) + \varphi((xb)a, y) + \varphi((ya)b, x) - \varphi((yb)a, x) + a \bullet \varphi(xy, b) - b \bullet \varphi(xy, a) - x \bullet \varphi(ya, b) + x \bullet \varphi(yb, a) + y \bullet \varphi(xa, b) - y \bullet \varphi(xb, a) - a \bullet (b \bullet \varphi(x, y)) + b \bullet (a \bullet \varphi(x, y)) - x \bullet (a \bullet \varphi(y, b)) + x \bullet (b \bullet \varphi(y, a)) - y \bullet (b \bullet \varphi(x, a)) + y \bullet (a \bullet \varphi(x, b))$

Cohomology? Variant 2

“CD derivations”

$$0 \to M \xrightarrow{\mathrm{d}^0} C^1(A, M) \xrightarrow{\mathrm{d}^1} C^3(A, M)$$

$\mathrm{d}^0(m)(x) = x \bullet m$
$\mathrm{d}^1(\varphi)(x, y, a) =$
$\varphi((xy)a) - a \bullet \varphi(xy) - y \bullet \varphi(xa) + x \bullet \varphi(ya) + y \bullet (a \bullet \varphi(x)) - x \bullet (a \bullet \varphi(y))$

Cohomology? Variant 3

By analogy with the Chevalley–Eilenberg

$$C^1(A,M) \xrightarrow{\mathrm{d}} C^3(A,M) \xrightarrow{\mathrm{d}} C^5(A,M) \xrightarrow{\mathrm{d}} \dots$$
$$C^2(A,M) \xrightarrow{\mathrm{d}} C^4(A,M) \xrightarrow{\mathrm{d}} C^6(A,M) \xrightarrow{\mathrm{d}} \dots$$

$$\begin{aligned}
&\mathrm{d}(\varphi)(x, y, a_1, \dots, a_n) \\
&\quad = \sum_{i=1}^{n} (-1)^i \Big(\varphi((xy)a_i, a_1, \dots, \widehat{a_i}, \dots, a_n) + a_i \bullet \varphi(xy, a_1, \dots, \widehat{a_i}, \dots, a_n) \\
&\qquad - x \bullet \varphi(ya_i, a_1, \dots, \widehat{a_i}, \dots, a_n) + y \bullet \varphi(xa_i, a_1, \dots, \widehat{a_i}, \dots, a_n) \\
&\qquad - x \bullet (a_i \bullet \varphi(y, a_1, \dots, \widehat{a_i}, \dots, a_n)) + y \bullet (a_i \bullet \varphi(x, a_1, \dots, \widehat{a_i}, \dots, a_n)) \Big) \\
&+ \sum_{1 \le i < j \le n} (-1)^{i+j+n+1} \Big(\varphi((xa_i)a_j, y, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n) - \varphi((xa_j)a_i, y, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n) \\
&\qquad - \varphi((ya_i)a_j, x, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n) + \varphi((ya_j)a_i, x, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n) \\
&\qquad + a_i \bullet (a_j \bullet \varphi(x, y, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n)) - a_j \bullet (a_i \bullet \varphi(x, y, a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n)) \Big)
\end{aligned}$$

Second CD cohomology

For any Lie algebra L and an L-module M, $\mathrm{H}^2(L, M) \subseteq \mathrm{H}^2_{CD}(L, M)$.

Conjecture 1

($p \neq 2, 3$) For any simple finite-dimensional Lie algebra L,

$$\mathrm{H}^2_{CD}(L, K) = \mathrm{H}^2(L, K).$$

Conjecture 2 ("Second CD Whitehead lemma")

($p = 0$) For any simple finite-dimensional Lie algebra L, and any finite-dimensional L-module M,

$$\mathrm{H}^2_{CD}(L, M) = 0.$$

Supported by computations in GAP.

Further questions

1) How "far" a CD algebra can be from Lie algebras? Describe CD algebras A with $LZ(A) = Z(A)$.

2) Study free CD algebras. Are they central extensions of free Lie algebras?

3) Study "CD speciality".

4) Study representations of CD algebras. An analog of the Ado theorem?

5) CD algebras without conditions of commutativity or anticommutativity?

That's all. Thank you.