A commutative 2-cocycles approach to classification of simple Novikov algebras

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What is a Novikov algebra?

(i) Left-symmetric (aka Vinberg, pre-Lie, chronological) identity:

$$x(yz) - (xy)z = y(xz) - (yx)z$$

(ii)

$$(xy)z = (xz)y$$

Equivalently:

$$[L_x, L_y] = L_{[x,y]}$$
$$[R_x, R_y] = 0$$

where $L_x(a) = xa$, $R_x(a) = ax$.

Origin

- Integrability of dynamical systems (Gelfand & Dorfman).
- Poisson brackets of hydrodynamic type (Balinsky & Novikov).

Crucial fact

Left-symmetricity $\Rightarrow$ Lie-admissibility:

$$[x, y] = xy - yx$$

satisfies the Jacobi identity.

Classification of finite-dimensional simple Novikov algebras over an algebraically closed field

Zelmanov (1987) $p = 0$: there are no non-trivial algebras.
Osborn (1992) $p > 2$: for any such non-trivial algebra A,

$$A^{(-)} \simeq W_1(n).$$

Xu (1996) $p > 2$: described completely.

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Reminder: Zassenhaus algebra

$W_1(n) = \langle e_{-1}, e_0, e_1, \ldots, e_{p^n-2} \rangle$
$[e_i, e_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i} \right) e_{i+j}.$

2-sided Alia algebras (Dzhumadil'daev, 2009)

Alia = **A**nti **Li**e-**a**dmissible

$$[x, y]z + [z, x]y + [y, z]x = 0$$
$$z[x, y] + y[z, x] + x[y, z] = 0$$

commutative
Lie
Novikov
LR

$\Rightarrow$ 2-sided Alia $\Rightarrow$ Lie-admissible

What are commutative 2-cocycles?

An algebra A is 2-sided Alia iff $L = A^{(-)}$ is a Lie algebra and multiplication in A is given by

$$xy = [x, y] + \varphi(x, y)$$

where $\varphi : L \times L \longrightarrow L$ is a **commutative** 2-**cocycle** on L, i.e.:

1. φ is symmetric
2. $\varphi([x, y], z) + \varphi([z, x], y) + \varphi([y, z], x) = 0$

$Z^2_{comm}(L) =$ the space of all K-valued commutative 2-cocycles on L.

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Theorem (Dzhumadil'daev & Zusmanovich, 2010)

A finite-dimensional simple Lie algebra over an algebraically closed field, $p \neq 2, 3$, possesses nonzero commutative 2-cocycles iff it is isomorphic to $sl(2)$ or $W_1(n)$.

$\dim Z^2_{comm}(sl(2)) = 5$.
$Z^2_{comm}(W_1(n)) \simeq O_1(n)^*$.

A subtle point

$p = 3$.

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Question

$p = 2?$

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That's all. Thank you.

Slides at `http://justpasha.org/math/coimbra.pdf`