On contact brackets on the tensor product

Pasha Zusmanovich

University of Ostrava

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Poisson brackets

A associative commutative algebra with unit, with the skew-symmetric bracket $[\cdot,\cdot]$

$$[ab,c] = [a,c]b + [b,c]a$$
$$[[a,b],c] + [[c,a],b] + [[b,c],a] = 0$$

A paradigmatic example

D, F derivations of A

$$[a,b] = D(a)F(b) - F(a)D(b)$$

Fact

For any Poisson brackets $[\cdot,\cdot]_A$ on A, and $[\cdot,\cdot]_B$ on B, there is a Poisson bracket $[\cdot,\cdot]$ on $A \otimes B$ extending $[\cdot,\cdot]_A$ and $[\cdot,\cdot]_B$.

Proof

$[a \otimes b, a' \otimes b'] = [a,a']_A \otimes bb' + aa' \otimes [b,b']$
(tensor product of Poisson algebras).

Contact brackets

A associative commutative algebra with unit, with the skew-symmetric bracket $[\,\cdot\,,\cdot\,]$

$$[ab,c]=[a,c]b+[b,c]a+[c,1]ab$$
$$[[a,b],c]+[[c,a],b]+[[b,c],a]=0$$

A paradigmatic example

D a derivation of A

$$[a,b]=D(a)b-D(b)a$$

Question (Martínez–Zelmanov 2019)

Does for any Poisson bracket $[\,\cdot\,,\cdot\,]_A$ on A, and contact bracket $[\,\cdot\,,\cdot\,]_B$ on B, there exist a contact bracket $[\,\cdot\,,\cdot\,]$ on $A\otimes B$ extending $[\,\cdot\,,\cdot\,]_A$ and $[\,\cdot\,,\cdot\,]_B$?

Answer

Yes & No.

“No” generally: a “minimal” example

$A = K[x,y]/(x^2,y^2)$ with Poisson bracket $[x,y]_A = xy$

$B = K[x]/(x^2)$ with contact bracket $[1,x]_B = x$

How to describe structures on the tensor product?

Let one of A, B is finite-dimensional. Then

$$\mathrm{Hom}_K(A \otimes B \otimes A \otimes B) \simeq \mathrm{Hom}_K(A \otimes A, A) \otimes \mathrm{Hom}_K(B \otimes B, B),$$

so any contact bracket on $A \otimes B$ is of the form

$$[a \otimes b, a' \otimes b'] = \sum_i f_i(a, a') \otimes g_i(b, b')$$

We deal with multiple conditions of the form $\sum_i S(f_i) \otimes T(g_i) = 0$ for some linear operators S and T. For example, symmetrizing and substituting 1 in the "contact condition" for this bracket, we get

$$\sum_i \Big(f_i(a, a'')a' - f_i(a', a'')a \Big) \otimes \Big(g_i(b, b'')b' - g_i(b', b'')b \Big) = 0.$$

As $\mathrm{Ker}(S \otimes T) \simeq \mathrm{Ker}(S) \otimes \cdot + \cdot \otimes \mathrm{Ker}(T)$, we may "partition" the set of indices i such that either $S(f_i) = 0$, or $T(g_i) = 0$.
The lemma on the next slide tells that we may do that simultaneously for two such conditions.

An old lemma and an application

An old lemma (Zusmanovich 2005)

Let U, W be two vector spaces, $S, S' \in \mathrm{Hom}(U, \cdot\,)$, $T, T' \in \mathrm{Hom}(W, \cdot\,)$. Then

$$\begin{aligned}\mathrm{Ker}(S \otimes T) \cap \mathrm{Ker}(S' \otimes T') & \\ \simeq & (\mathrm{Ker}\, S \cap \mathrm{Ker}\, S') \otimes W \\ + & \mathrm{Ker}\, S \otimes \mathrm{Ker}\, T' \\ + & \mathrm{Ker}\, S' \otimes \mathrm{Ker}\, T \\ + & U \otimes (\mathrm{Ker}\, T \cap \mathrm{Ker}\, T').\end{aligned}$$

An example of application

$$\mathcal{K}^{-}\big(A \otimes K[x]/(x^2)\big) \simeq \mathcal{K}^{-}(A) \oplus \mathcal{K}^{-}(A) \oplus \mathrm{Der}(A) \oplus A,$$

where $\mathcal{K}^{-}(A)$ is the space of skew-symmetric linear maps on A with the "contact condition".

"Yes" for polynomial(-like) algebras

For some algebras, each contact bracket is the sum of paradigmatic examples, i.e., is of the form

$$\sum_i \Big(D_i(a)F_i(b) - D_i(b)F_i(a) \Big) + D(a)b - D(b)a.$$

Examples: $K[x_1, \ldots, x_n]$ and $K[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$.

For such algebras A and B, any contact brackets on A and B are extended to a contact bracket on $A \otimes B$.

The text is available as
https://web.osu.cz/∼Zusmanovich/papers/tensprodcontact.pdf

That's all. Thank you.