# Approximability of Lie groups

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# Zilber's program

Treatment of quantum mechanics (and foundations of physics in general) from the point of view of model theory. The main goal is to (properly) handle infinities in physics.

# Zilber's program

Some highlights:

- Physics practice (finite computations) dictate that it should be treated in the scope of a first-order, and not higher order, theory.
- Instead of analytic continuity consider continuity in terms of Zariski topology.
- Physics should be based on mathematical structures which are categorical (for example, $\mathbb{C}$ qualifies, while $\mathbb{R}$ does not). Reasons: such structures 1) exhibit homogeneity 2) allow notion of dimension.
- The physical principle "quantum mechanics applied to 'large' structures degenerates to classical mechanics" translates to mathematical principle of "finite approximation": 'large' finite structures are treated as infinite ones having first-order categorical theory.

# Some parallel developments

(All around 2013–2014)

- Chalons & Ressayre
  Phenomenological approach to quantum mechanics: mathematical objects exist up to an error in their transmission. Technical tool: predicate-like calculus of binary relations.
- Kapustin, Moldoveanu
  System of axioms, basing on category theory, and involving some Lie-algebraic structures, leading to Quantum Mechanics as the only possible theory. (Sort of) continuity is assumed, appearance of $\mathbb{C}$ follows from axioms.
- Kornyak
  "Quantum discrete dynamical systems". Discretize everything; instead of $\mathbb{R}$ or $\mathbb{C}$, use cyclotomic fields.

# Approximation according to Zilber

## Definition

A structure $A$ is approximated by a family of structures $\{B_i\}$ if there is a surjection $\prod_{\mathscr{U}} B_i$ (ultraproduct) $\to A$.

Compare with other (more traditional?) notions of approximation: *embedding* into direct product or ultraproduct.

## Question

Whether (say) $SO(3)$ is approximated by finite groups?

**No**, if finite groups are simple (Pillay, 2015).

# From ultraproducts to direct products

## Theorem (Bergman–Nahlus)

For any cardinal $\kappa > 2$, and any algebraic system $A$ consisting of more than one element, the following are equivalent:

(i) For any surjective homomorphism $f : \prod_{i\in\mathbb{I}} B_i \to A$, $|\,\mathbb{I}\,| < \kappa$, there is $i_0 \in \mathbb{I}$ such that $f$ factors through the canonical projection $\prod_{i\in\mathbb{I}} B_i \to B_{i_0}$.

(ii) For any surjective homomorphism $f : \prod_{i\in\mathbb{I}} B_i \to A$, there is a $\kappa$-complete ultrafilter $\mathscr{U}$ on $\mathbb{I}$ such that $f$ factors through the canonical homomorphism $\prod_{i\in\mathbb{I}} B_i \to \prod_{\mathscr{U}} B_i$.

## Corollary

Zilber's question is equivalent to: whether the *direct product* of finite groups can be mapped surjectively onto $SO(3)$?

# References

B. Zilber, *Perfect infinities and finite approximation*, Infinity and Truth (ed. C. Chong et al.), World Scientific, 2014, 199–223.

P. Zusmanovich, *On the utility of Robinson–Amitsur ultrafilters. I, II*, J. Algebra **388** (2013), 268–286; **466** (2016), 370–377; arXiv: 0911.5414, 1508.07496

That's all. Thank you.