Low-dimensional cohomology of current Lie algebras

Pasha Zusmanovich

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What is a Lie algebra?

anticommutativity: $[x, y] = -[y, x]$
Jacobi identity: $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$

Example: algebra of all $n \times n$ matrices under $[X, Y] = XY - YX$

Another (boring) example: *abelian* Lie algebra: $[x, y] = 0$

Another (interesting) example: $Der(A)$ for any algebra A

$D : A \to A$ is a *derivation of A* if
$D(ab) = D(a)b + aD(b)$ for any $a, b \in A$
$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$

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What is a current Lie algebra?

L - Lie algebra $\quad A$ - associative commutative algebra
$L \otimes A$
$[x \otimes a, y \otimes b] = [x, y] \otimes ab \quad$ for $x, y \in L$, $a, b \in A$.

What current Lie algebras are good for?

Kac-Moody algebras

$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C} t \frac{d}{dt} + \mathbb{C} z$

$[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y) Res \frac{df}{dt} g \; z$
for $x, y \in \mathfrak{g}$, $f, g \in \mathbb{C}[t, t^{-1}]$

$(\cdot , \cdot)$ - *symmetric invariant bilinear form* on $\mathfrak{g}$
$([x, y], z) = (x, [z, y])$
example: $(X, Y) = Tr(XY)$ on the matrix algebra

It is a *central extension* of the current Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$:
$0 \to Z \to (\text{central extension}) \to L \to 0$

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Modular semisimple Lie algebras

$S \otimes K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p) + 1 \otimes D$

What current Lie algebras are good for? (cont.)

Physics

Gauge theory: how point particles transform as they move along paths in spacetime:

spacetime = smooth manifold M, for example, a cylinder $S^1 \times \mathbb{R}$
transformation = element of a smooth (Lie) group G acting on M.

String theory:
string = (bunch of) loops
gauge group = loop group $\{$smooth functions $S^1 \to G\}$.

Loop algebra: $\{$smooth functions $S^1 \to L\} = L \otimes \mathbb{R}[t, t^{-1}]$

Quantum mechanics:
Phases are unobservable, so one considers representations of groups "up to a phase", i.e. *projective representations* = representations of a central extension.
Central extension of a loop algebra = Kac-Moody algebra.

What is a cohomology of Lie algebras?

$$C^0(L,M) \xrightarrow{d} C^1(L,M) \xrightarrow{d} C^2(L,M) \xrightarrow{d} C^3(L,M) \xrightarrow{d} \dots$$

$$C^n(L,M) = \{\text{skew-symmetric multilinear maps} : L \times \cdots \times L \to M\}$$

$$d : C^n(L,M) \to C^{n+1}(L,M)$$

$$\begin{aligned} & d\varphi(x_1, \dots, x_{n+1}) \\ &= \sum_{1 \le i < j \le n+1} (-1)^{i+j-1} \varphi([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{n+1}) \\ &+ \sum_{i=1}^{n+1} (-1)^i x_i \bullet \varphi(x_1, \dots, \widehat{x_i}, \dots, x_n) \end{aligned}$$

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Low-dimensional interpretations

- Abelian extensions
- Derivations
- Deformations

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  L - Lie algebra, M - L-module
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- Derivations
  derivations = 1-cocycles
  $D([x, y]) - [x, D(y)] + [y, D(x)] = 0$
  inner derivations = 1-coboundaries
  $D(x) = [x, a]$
  outer derivations = $H^1(L, L)$
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- Deformations
  L over $K \rightsquigarrow \mathcal{L}$ over $K((t))$
  $\{x, y\} = [x, y] + \varphi_1(x, y)t + \varphi_2(x, y)t^2 + \dots$
  "infinitesimal" deformations = $H^2(L, L)$
  obstructions to "integrability" = $H^3(L, L)$
  (Gerstenhaber, 1960s)

What cohomology of Lie algebras is good for? (cont.)

Combinatorial identities

$\mathcal{C} : C_0 \xrightarrow{d} C_1 \xrightarrow{d} C_2 \xrightarrow{d} C_3 \xrightarrow{d} \dots$

Euler–Poincaré characteristic $\chi(\mathcal{C}) = \sum_{n \geq 0} (-1)^n \dim C_n$

Euler–Poincaré principle: $\chi(\mathcal{C}) = \chi(H^*(\mathcal{C}))$.

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$$\prod_{n=1}^{\infty} (1 - t^n) = 1 + \sum_{n=1}^{\infty} (-1)^n (t^{\frac{3n^2-n}{2}} + t^{\frac{3n^2+n}{2}})$$

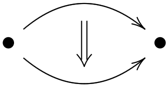
Euler, 1740s – conjectured

Garland & Lepowsky, 1975-1976 – proved using cohomology of some subalgebras of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

What cohomology of Lie algebras is good for? (cont.)

2-Lie algebras

"Categorified" Lie algebras = Lie-like structures on a category in the category of vector spaces (i.e. objects and morphisms are vector spaces)
Used in "higher gauge theory": how strings transform as they move along surfaces in spacetime:



Classified in terms of H^3 of ordinary Lie algebras.
(Baez & Co., 2002-2009)

What is known about (co)homology of some particular current Lie algebras?

H^* of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, $\mathfrak{g} \otimes \mathbb{C}[t]$, $\mathfrak{g} \otimes \mathbb{C}[t]/(t^n)$, etc.
(Feigin, Garland, Hanlon, Lepowsky, and others, 1975–1996)

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$H_*(gl(A)) \simeq \bigwedge(HC_*(A))$ and additive K-theory (Tsygan, 1983 and Loday–Quillen, 1984)

Cyclic (co)homology:
$HC^1(A) = \{\alpha : A \times A \to K \mid \alpha(ab, c) + \alpha(ca, b) + \alpha(bc, a) = 0\}$

How cyclic cohomology appears in current Lie algebras cohomology?

Let $\varphi \otimes \alpha \in Z^2(L \otimes A, K)$, $\varphi : L \times L \to K$, $\alpha : A \times A \to K$:

$$\begin{aligned} &\varphi([x,y],z) \otimes \alpha(ab,c) \\ +\,&\varphi([z,x],y) \otimes \alpha(ca,b) \\ +\,&\varphi([y,z],x) \otimes \alpha(bc,a) = 0 \end{aligned}$$

for any $x, y, z \in L$, $a, b, c \in A$.

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for any $x, y, z \in L$, $a, b, c \in A$.

But what for the general cocycle $\sum_{i \in I} \varphi_i \otimes \alpha_i \in Z^n(L \otimes A)$?

How to compute cohomology of "general" current Lie algebras?

Task: to "compute" $H^n(L \otimes A, M \otimes V)$.

$$
\begin{aligned}
Z^n(L \otimes A, M \otimes V) \subset Hom((L \otimes A)^{\otimes n}, M \otimes V) \\
\simeq Hom(L^{\otimes n}, M) \otimes Hom(M^{\otimes n}, V)
\end{aligned}
$$

Various symmetrizations of the cocycle equation $d\Phi = 0$ lead to conditions $(S \otimes T)\Phi = 0$, where $S \in Hom(Hom(L^{\otimes n}, M))$, $T \in Hom(Hom(A^{\otimes n}, V))$.

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For example: fully symmetrize the cocycle equation $d\Phi(x_1 \otimes a_1, x_2 \otimes a_2, x_3 \otimes a_3) = 0$ with respect to x_i's, where $\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i$, $\varphi_i : L \times L \to M$, $\alpha_i : A \times A \to V$:

$$\sum_{i \in I} \Big((x_1 \bullet \varphi_i(x_2, x_3) + x_1 \otimes \varphi_i(x_3, x_2) + x_2 \bullet \varphi_i(x_1, x_3) + x_2 \otimes \varphi_i(x_3, x_1) + x_3 \bullet \varphi_i(x_1, x_2) + x_3 \otimes \varphi_i(x_2, x_1)) \otimes (-a_1 \bullet \alpha_i(a_2, a_3) + a_2 \bullet \alpha_i(a_1, a_3) - a_3 \bullet \alpha_i(a_1, a_2)) \Big) = 0$$

What is known about (co)homology of “general” current Lie algebras?

Nice formulae:

- $H^1(L \otimes A, M \otimes V) \simeq H^1(L, M) \otimes V + Hom_L(L, M) \otimes Der(A, V)$ (Zusmanovich, 2005)
- $H^2(L \otimes A, K) \simeq H^2(L, K) \otimes A^* + B(L) \otimes HC^1(A)$ (Haddi, 1992)
  (both assuming $[L, L] = L$)
- If $\mathfrak{g}$ is a simple Lie algebra ($p = 0$), then $H^2(\mathfrak{g} \otimes A, \mathfrak{g} \otimes A) \simeq Har^2(A, A)$ (Cathelineau, 1987)
- If $W_1(n)$ is the modular Zassenhaus algebra, then

$$H^2(W_1(n) \otimes A, W_1(n) \otimes A)$$
$$\simeq H^2(W_1(n), W_1(n)) \otimes A + Der(A) + Der(A) + Har^2(A, A)$$

  (Zusmanovich, 2003)
- $H^3(\mathfrak{g} \otimes A, K) \simeq HC^2(A)$ or $HD^2(A)$ (Cathelineau, 1987)

Application to structure theory of modular Lie algebras

A bit more about $H^2(W_1(n) \otimes A, W_1(n) \otimes A)$...

$W_1(n) = \langle e_{-1}, e_0, e_1, \dots, e_{p^n-2} \rangle$,

$$[e_i, e_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i} \right) e_{i+j}.$$

Description of modular semisimple Lie algebras with a solvable maximal subalgebra (Weisfeiler, 1984): filtered deformations of $W_1(n) \otimes K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p) + 1 \otimes D$.

Filtered deformation: (graded algebra) $\rightsquigarrow$ (filtered algebra)
filtered algebra: $\mathcal{L} = \mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots$
associated graded algebra: $L = \bigoplus_{i \geq -1} \mathcal{L}_i / \mathcal{L}_{i+1}$

What is known about (co)homology of "general" current Lie algebras? (cont.)

Ugly formulae: some part (not all!) of $H^2(L \otimes A, M \otimes V)$ is isomorphic to

$$H^2(L,M) \otimes V + H^2_M(L) \otimes \frac{Hom(A,V)}{V \oplus Der(A,V)} + \mathscr{H}(L,M) \otimes Der(A,V) + \mathscr{B}(L,M) \otimes \frac{Har^2(A,V)}{\mathscr{P}_+(A,V)}$$
$$+ C^2(L,M)^L \otimes \mathscr{P}_+(A,V) + \mathscr{X}(L,M) \otimes \frac{\mathscr{A}(A,V)}{\mathscr{P}_+(A,V)} + \mathscr{T}(L,M) \otimes \frac{D(A,V)}{Der(A,V)}$$
$$+ Poor_-(L,M) \otimes \frac{S^2(A,V)}{Hom(A,V) + D(A,V) + Har^2(A,V) + \mathscr{A}(A,V)}$$

where:
$d^{[\,]}\varphi(x,y,z) = \varphi([x,y],z) + \curvearrowright;$
$\wp\alpha(a,b,c) = \alpha(ab,c) + \curvearrowright;$
$D\alpha(a,b,c) = a \bullet \alpha(b,c) + \curvearrowright;$
$\mathscr{B}(L,M) = \{\varphi \in C^2(L,M) \mid \varphi([x,y],z) + z \bullet \varphi(x,y) = 0; d^{[\,]}\varphi(x,y,z) = 0\};$
$Q^2(L,M) = \{d\psi \mid \psi \in Hom(L,M); x \bullet \psi(y) = y \bullet \psi(x)\};$
$H^2_M(L) = (Z^2(L,M^L) + Q^2(L,M))/Q^2(L,M);$
$\mathscr{J}(L,M) = \{\varphi \in C^2(L,M) \mid \varphi(x,y) = \psi([x,y]) - \frac{1}{2}x \bullet \psi(y) + \frac{1}{2}y \bullet \psi(x) \text{ for } \psi \in Hom(L,M)\};$
$\mathscr{H}(L,M) = (\mathscr{K}(L,M) + \mathscr{J}(L,M))/\mathscr{J}(L,M).$
$\mathscr{X}(L,M) = \{\varphi \in C^2(L,M) \mid 2\varphi([x,y],z) = z \bullet \varphi(x,y); \varphi([x,y],z) = \varphi([z,x],y)\};$
$\mathscr{T}(L,M) = \{\varphi \in C^2(L,M) \mid 3\varphi([x,y],z) = 2z \bullet \varphi(x,y); \varphi([x,y],z) = \varphi([z,x],y)\};$
$Poor_-(L,M) = \{\varphi \in C^2(L,M^L) \mid \varphi([L,L],L) = 0\};$
$D(A,V) = \{\beta \in Hom(A,V) \mid \beta(abc) = a \bullet \beta(bc) - bc \bullet \beta(a) + \curvearrowright\};$
$\mathscr{P}_+(A,V) = \{\alpha \in S^2(A,V) \mid \alpha(ab,c) = a \bullet \alpha(b,c) + b \bullet \alpha(a,c)\};$

$\mathscr{A}(A,V) = \{\alpha \in S^2(A,V) \mid 2D\alpha = \wp\alpha\}.$

How to (methodically) compute cohomology of “general” current Lie algebras?

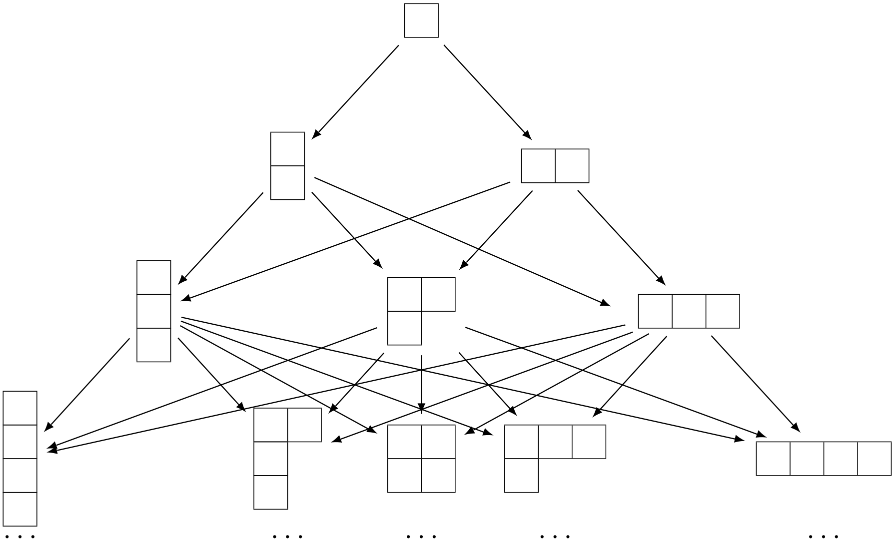
Cauchy formula:

$$\bigwedge^n (L \otimes A) \simeq \bigoplus_{\lambda \vdash n} Y_\lambda(L) \otimes Y_{\lambda^\sim}(A)$$

Y_λ - *Schur functor* associated with the Young diagram λ.
Examples:
$Y_{\ydiagram{1,1,1}} = \frac{1}{3!} \sum_{\sigma \in S_3} (-1)^\sigma \sigma$
$Y_{\ydiagram{3}} = \frac{1}{3!} \sum_{\sigma \in S_3} \sigma$
$Y_{\ydiagram{2,1}} = \frac{1}{3}(e + (12) - (13) - (123))$

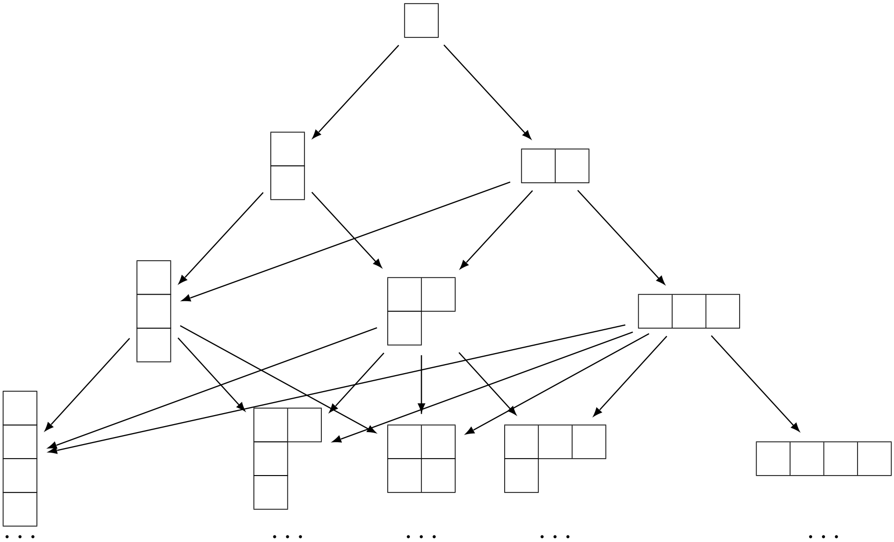
$\lambda^\sim$ - obtained from λ by interchanging rows and columns

A spectral sequence...



each Young diagram λ represents $Hom(Y_\lambda(L), M) \otimes Hom(Y_\lambda(A), V)$

A spectral sequence...



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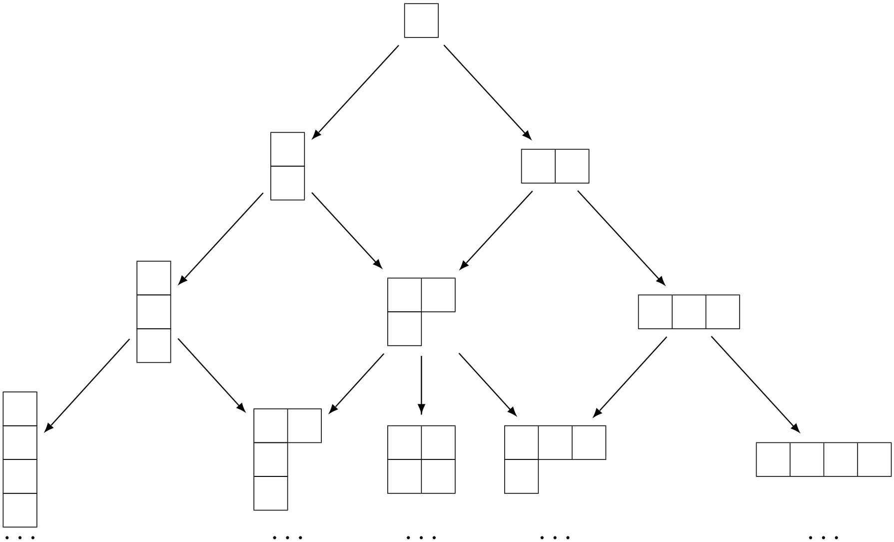
A spectral sequence... (abelian L)

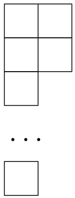


each Young diagram λ represents $Hom(Y_\lambda(L), M) \otimes Hom(Y_\lambda(A), V)$

A spectral sequence (cont.)

Which arrows do not vanish?

- going from "right" to "left"
- the source Young diagram included into the target Young diagram (the only case when L is abelian)
- the target Young diagram is of the shape:



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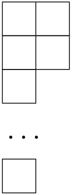
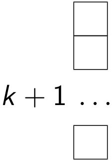


Filtration: $F^k C^* =$ “closure” under non-vanishing arrows of



That's all. Thank you.

Slides at `http://justpasha.org/math/iceland-2009.pdf` .