On the utility of Robinson–Amitsur ultrafilter

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A theorem from 1960s

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Reminder 1

An (associative) ring R is called *prime* if one of the following equivalent conditions holds:

(i) $\forall\, I, J \triangleleft R \;\; I, J \neq 0 \Rightarrow IJ \neq 0$;

(ii) $\forall a, b \in R, a, b \neq 0\, \exists x \in R : axb \neq 0$.

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Reminder 2

A ring R is called *division ring* if
$\forall a, b \in R\, \exists x, y \in R : ax = b\,\&\, ya = b$.

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An *ultrafilter* $\mathcal{U}$ on a set $\mathbb{I}$ is a set of subsets of $\mathbb{I}$ satisfying the following conditions:

1. $\varnothing \notin \mathcal{U}$
2. $\mathbb{X}, \mathbb{Y} \in \mathcal{U} \Rightarrow \mathbb{X} \cap \mathbb{Y} \in \mathcal{U}$
3. $\mathbb{X} \in \mathcal{U}, \mathbb{X} \subset \mathbb{Y} \Rightarrow \mathbb{Y} \in \mathcal{U}$
4. $\forall \mathbb{X} \subset \mathbb{I} \quad (\mathbb{X} \in \mathcal{U}) \vee (\mathbb{I} \backslash \mathbb{X} \in \mathcal{U})$ (maximality)

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Examples

A *principal ultrafilter*: $\{\mathbb{X} \subset \mathbb{I} \mid i \in \mathbb{X}\}$ for some $i \in \mathbb{I}$.
A *cofinite filter*: $\{\mathbb{X} \subset \mathbb{I} \mid \mathbb{I} \backslash \mathbb{X} \text{ is finite}\}$.

The proof (cont.)

Fact

Any set $\mathcal{S}$ satisfying:

$$\mathbb{X}, \mathbb{Y} \in \mathcal{S} \Rightarrow \mathbb{X} \cap \mathbb{Y} \neq \varnothing$$

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Reminder

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{I}$. An *ultraproduct* of a set of rings $\{A_i\}_{i \in \mathbb{I}}$ is the ring

$$\prod_{\mathcal{U}} A_i = \Big(\prod_{i \in \mathbb{I}} A_i \Big) \Big/ \{f \mid \{i \in \mathbb{I} \mid f_i = 0\} \in \mathcal{U}\}.$$

If $A_i \simeq A$, we have an *ultrapower* $A^{\mathcal{U}}$.

The proof (cont.)

Łoś' theorem

For any first-order sentence φ,

$$\prod_{\mathcal{U}} A_i \models \varphi \quad \Leftrightarrow \quad \{i \in \mathbb{I} \mid A_i \models \varphi\} \in \mathcal{U}.$$

In particular,

$$Th\Big(\prod_{\mathcal{U}} A_i\Big) \supseteq \bigcap_{i \in \mathbb{I}} Th(A_i)$$

and

$$A^{\mathcal{U}} \equiv A.$$

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Define $\mathcal{S} = \{\{i \in \mathbb{I} \mid f_i \neq 0\} \mid f \in R, f \neq 0\}$.
Which (ultra)filter properties it satisfies?

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   $\mathbb{X} = \{i \in \mathbb{I} \mid f_i \neq 0\}$, $0 \neq f \in R$
   $\mathbb{Y} = \{i \in \mathbb{I} \mid g_i \neq 0\}$, $0 \neq g \in R$
   Since R is prime, there is $x \in R$ such that $fxg \neq 0$, and

$$\begin{aligned}\mathbb{Z} = \{i \in \mathbb{I} \mid (fxg)_i = f_i x_i g_i \neq 0\} &\subset \{i \in \mathbb{I} \mid f_i \neq 0\} = \mathbb{X} \\ &\subset \{i \in \mathbb{I} \mid g_i \neq 0\} = \mathbb{Y}.\end{aligned}$$

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$\mathcal{S}$ contained in some ultrafilter $\mathcal{U}$.

$$R = R \Big/ \left(R \cap \{f \in \prod_{i \in \mathbb{I}} A_i \mid \{i \in \mathbb{I} \mid f_i = 0\} \in \mathcal{U}\} \right) \subseteq \prod_{\mathcal{U}} A_i .$$

A generalization

Theorem ("Robinson–Amitsur for algebras")

If a prime (nonassociative) algebra R embeds in the direct product $\prod_{i \in \mathbb{I}} A_i$, then R embeds in an ultraproduct $\prod_{\mathcal{U}} A_i$.

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Definition

A (nonassociative) algebra is called *prime* if one of the following equivalent conditions holds:

(i) $\forall\, I, J \triangleleft R \;\; I, J \neq 0 \Rightarrow IJ \neq 0$;

(ii) $\forall a, b \in R, a, b \neq 0$ there is a nonzero word in elements of R containing a, b.

A question

Does the converse is true (at least in the associative case)? Suppose that for an algebra R the following holds: for any set of algebras $\{A_i\}_{i\in\mathbb{I}}$, if R embeds in the direct product $\prod_{i\in\mathbb{I}} A_i$, then R embeds in an ultraproduct $\prod_{\mathcal{U}} A_i$. Does this imply that R is prime? that R satisfies any other natural structural condition?

Birkhoff meets Robinson–Amitsur

Birkhoff's theorem

A class of algebras is a variety iff it is closed under subalgebras, quotients and direct products.

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Corollary (Birkhoff + Robinson–Amitsur)

A prime relatively free algebra $\mathscr{F}$ in a variety $Var(A)$ embeds in an ultrapower $A^{\mathcal{U}}$.

Proof. By Birkhoff's theorem, $\mathscr{F} = B/I$ for some $B \subseteq A^{\mathbb{I}}$ and $I \triangleleft B$. By the universal property of $\mathscr{F}$, it embeds in B and hence in $A^{\mathbb{I}}$. Apply Robinson–Amitsur for algebras.

Criterion for absence of non-trivial identities of algebras

Theorem

For an algebra R over a field K belonging to one of the following variety of algebras: all algebras, associative algebras, or Lie algebras, the following is equivalent:

(i) R does not satisfy a nontrivial identity;

(ii) A free algebra embeds in an ultrapower of R;

(iii) A free algebra embeds in an algebra elementary equivalent to R.

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(ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are trivial.
(i) $\Rightarrow$ (ii) "almost" follows from "Birkhoff + Robinson–Amitsur".
Two issues:

- Whether a free algebra is prime?
- "Birkhoff + Robinson–Amitsur" gives embedding on the level of K-algebras, not of $K^{\mathcal{U}}$-algebras.

In which varieties free algebras are prime?

name	**identities**		**why?**
all algebras		yes	no zero divisors
associative	$(xy)z = x(yz)$	yes	
Lie	$xy = -yx$ $(xy)z + (zx)y + (yz)x = 0$	yes	Shirshov–Witt theorem
alternative	$(xx)y = x(xy)$ $(xy)y = x(yy)$	no	explicit example
Jordan	$xy = yx$ $(xy)(xx) = x(y(xx))$	no	explicit example

K-algebras vs. $K^{\mathcal{U}}$-algebras

"Birkhoff + Robinson–Amitsur" establishes embedding (under appropriate conditions) of K-algebras:

$$\mathscr{F} \subseteq A^{\mathcal{U}}.$$

To be able to apply Łoś' theorem, one needs embedding of $K^{\mathcal{U}}$-algebras:

$$K^{\mathcal{U}}\mathscr{F} \subseteq A^{\mathcal{U}}.$$

By the universal property of the tensor product, we have a surjection

$$\mathscr{F} \otimes_K K^{\mathcal{U}} \to K^{\mathcal{U}}\mathscr{F},$$

but, generally, this is far from being an isomorphism. This is so, however, if $\mathscr{F}$ does not have commutative subalgebras of dimension > 1, in particular, for absolutely free, free associative and free Lie algebras.

“Robinson–Amitsur for groups”

Theorem

If a group G, all whose abelian subgroups are cyclic either of prime order, or of infinite order, embeds in the direct product $\prod_{i\in\mathbb{I}} F_i$ of groups, then G embeds in an ultraproduct $\prod_{\mathcal{U}} F_i$.

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Criterion for absence of non-trivial identities of groups

Theorem

For a group G the following is equivalent:

(i) G does not satisfy a nontrivial identity;

(ii) A nonabelian free group embeds in an ultrapower of G;

(iii) A nonabelian free group embeds in a group elementary equivalent to G.

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Question

Semigroups?

Application: PI

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If A, B are PI, then $A \otimes B$ is PI.

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If $A = M_n(K)$, and B is PI, then $A \otimes B$ is PI.

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Theorem

If A is finite-dimensional, B is PI, then $A \otimes B$ is PI.

Proof. Suppose the contrary. Then a free associative algebra of exponential growth embeds in

$$(A \otimes_K B)^{\mathcal{U}} \simeq (A \otimes_K K^{\mathcal{U}}) \otimes_{K^{\mathcal{U}}} B^{\mathcal{U}}$$

of polynomial growth, a contradiction.

Application: algebras with same identities

Theorem (Razmyslov, 1982)

If $\mathfrak{g}_1, \mathfrak{g}_2$ are finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 0, then $Var(\mathfrak{g}_1) = Var(\mathfrak{g}_2)$ iff $\mathfrak{g}_1 \simeq \mathfrak{g}_2$.

Theorem (Drensky–Racine, 1992)

Ditto for finite-dimensional simple Jordan algebras.

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Joint proof. Using the fact that in these classes of algebras primeness is equivalent to simplicity, and by the embedding machinery above, reduced to the case where $\mathfrak{g}_1 \subseteq \mathfrak{g}_2$, which follows easily from the known structural results.

Application: Tarski's monsters

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Growth sequence

Number of generators of $G \times \cdots \times G$ (n times).
Wise (2002): groups with growth sequence equal to 2.

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Theorem

(i) A Tarski's monster of type p does not satisfy any nontrivial identity except $x^p = 1$ and its consequences, iff it has infinite relative girth.

(ii) A Tarski's monster of type ∞ does not satisfy any nontrivial identity iff it has infinite girth.

(iii) Growth sequence of Tarski's monsters satisfying conditions (i) or (ii), is 2.

Speculations

Lie-algebraic analogs of Tarski's monsters

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Tits' alternative

Tits (1972): a linear group contains either a solvable subgroup of finite index, or a nonabelian free subgroup.
Platonov (1967): a linear group which satisfies a nontrivial identity, contains a solvable subgroup of finite index.

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Platonov (1967): a linear group which satisfies a nontrivial identity, contains a solvable subgroup of finite index.

Jacobson's problem

In a Lie p-algebra, does $x^{p^{n(x)}} = x$ imply abelianity?
Weaker question: does it imply a nontrivial identity?

That's all. Thank you.

Based on arXiv:0911.5414 .
Slides at `http://justpasha.org/math/iceland-2010.pdf` .