Lie algebras with few subalgebras, cohomology and dual operads

Pasha Zusmanovich

Tallinn University of Technology

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Lie algebras with given properties of subalgebras

Question

Study such Lie algebras.

More precise questions

- Lie algebras with "few" subalgebras.
- Lie algebras with given properties of the lattice of subalgebras.
- Minimal non-$\mathscr{P}$ Lie algebras, for some "nice" $\mathscr{P}$.

Lie algebras all whose proper subalgebras are 1-dimensional

Exercise

Over an algebraically closed field, every such finite-dimensional Lie algebra is 2-dimensional.

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Proof: Either follows from A. Premet (1987), or by classification theory.

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What about nonperfect fields and/or $p = 2, 3$?

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A very hard open question

What about such infinite-dimensional Lie algebras (analogs of Tarski's monsters)?

Lie-algebraic analogs of Tarski's monsters?

Possible approaches:

- First- (or higher?) order theory.
- Girth.
- (Absence of) identities.

Lie algebras all whose proper subalgebras are abelian

Exercise

(i) Over an algebraically closed field, every such nonabelian finite-dimensional Lie algebra is either 2-dimensional nonabelian, or 3-dimensional nilpotent (Heisenberg).

(ii) Describe the structure of such nonsimple Lie algebras over any field.

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Theorem

Over a perfect field of characteristic 0 or $p > 3$, there are no such nonabelian simple finite-dimensional Lie algebra of types B-D, G_2 and F_4.
Proof: By inspection of (associative) division algebras with involution.

An open question

What about other types?

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A curious connection

A.M. Vinogradov (2012): "Assembling Lie algebras from Lieons".

Lie algebras all whose proper subalgebras are solvable

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Theorem

Over an algebraically closed field of characteristic $p > 3$, every such nonsolvable finite-dimensional Lie algebra is an abelian extension of sl(2) by modules of some specific type (decomposable into the direct sum of no more than 2 components, with each indecomposable component of length ≤ 3, etc.).

Proof (modulo N. Jacobson (1958), A. Rudakov & I. Shafarevich (1967), J. Schue (1969)):
Cohomological juggling (computing Ext^1, H^2, etc.) with sl(2)-modules.

Lie algebras with a maximal solvable subalgebra

Exercise

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Theorem

Over an algebraically closed field of characteristic $p > 5$, every such semisimple finite-dimensional Lie algebra is isomorphic to an algebra of the form

$$S \otimes K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p) + \text{“some tail of derivations”},$$

where S is isomorphic to sl(2), or the Zassenhaus algebra $W_1(n)$.

Proof (modulo B. Weisfeiler (1984)):
Computing deformations of algebras of this kind.

Current Lie algebras

L is a Lie algebra
A is an associative commutative algebra

A **current Lie algebra** is a vector space $L \otimes A$ under the bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

where $x, y \in L$, $a, b \in A$.

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Kac-Moody algebras

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}t\frac{d}{dt} + \mathbb{C}z$$

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y) Res\frac{df}{dt}g\ z$$

where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, $x, y \in \mathfrak{g}$, $f, g \in \mathbb{C}[t, t^{-1}]$, $(\cdot, \cdot)$ is the Killing form on $\mathfrak{g}$.

(Co)homology of current Lie algebras

Question

What can be said about it?

Answer

A lot:

B. Feigin (1970–1990s),
H. Garland & J. Lepowsky (1976),
B. Feigin & B. Tsygan (1983–1984),
J.-L. Loday & D. Quillen (1984),
P. Hanlon (1986), ...

but ...

How to "compute" (co)homology of current Lie algebras "in general"?

Cauchy formula:

$$\bigwedge^n (L \otimes A) \simeq \bigoplus_{\lambda \vdash n} Y_\lambda(L) \otimes Y_{\lambda^\sim}(A)$$

Y_λ is a *Young symmetrizer* associated with the Young diagram λ.

Examples:

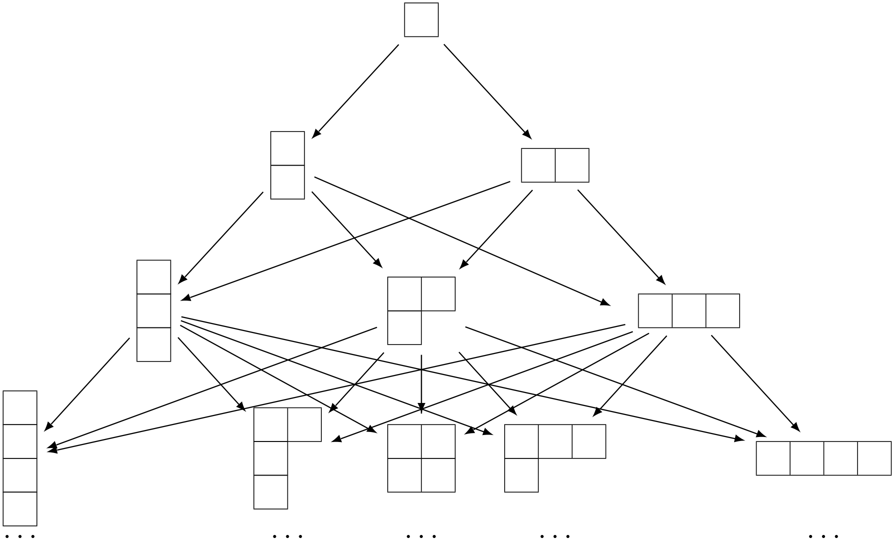
$Y_{\yng(1,1,1)} = \frac{1}{3!}\sum_{\sigma \in S_3} (-1)^\sigma \sigma$

$Y_{\yng(3)} = \frac{1}{3!}\sum_{\sigma \in S_3} \sigma$

$Y_{\yng(2,1)} = \frac{1}{3}(e + (12) - (13) - (123))$

$\lambda^\sim$ is obtained from λ by interchanging rows and columns

How Young symmetrizers interact with the differential?



each Young diagram λ represents $Hom(Y_\lambda(L), M) \otimes Hom(Y_\lambda(A), V)$

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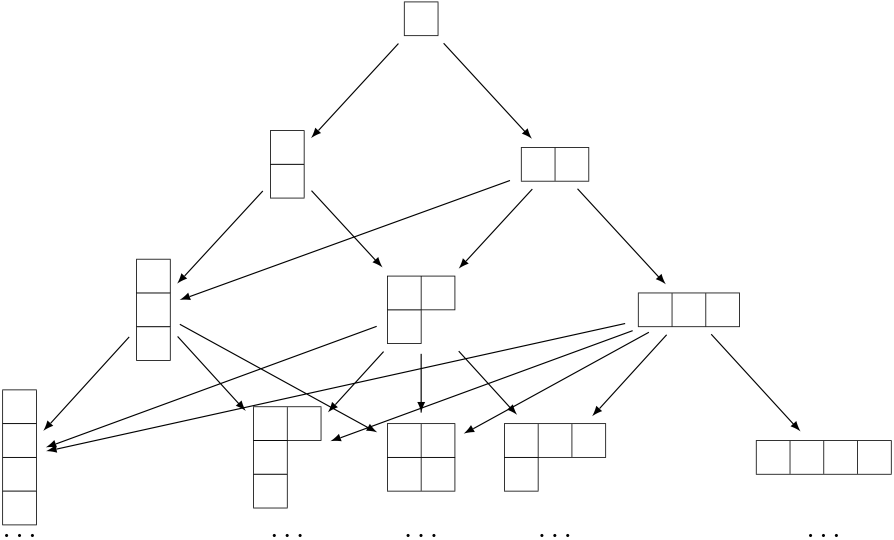

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Lie algebras coming from Koszul dual operads

Fact

If:

A is an algebra over a binary quadratic operad $\mathscr{P}$,

B is an algebra over the Koszul dual operad $\mathscr{P}^!$,

then:

$A \otimes B$ carries a Lie algebra structure under the bracket

$$[a \otimes b, a' \otimes b'] = aa' \otimes bb' - a'a \otimes b'b,$$

where $a, a' \in A,\ b, b' \in B$.

Examples

operad	**dual operad**	**Lie algebras**
Lie	associative commutative	current Lie algebras
associative	associative	$\mathrm{gl}_n(A)$
left Novikov	right Novikov	... stay tuned ...

Novikov algebras and their affinizations

left Novikov algebra: $[L_x, L_y] = L_{[x,y]}$; $[R_x, R_y] = 0$
right Novikov algebra: $[R_x, R_y] = R_{[x,y]}$; $[L_x, L_y] = 0$
where $L_x(a) = xa$, $R_x(a) = ax$

Affinization of a left Novikov algebra

$$N \otimes \mathbb{C}[t, t^{-1}]$$

$$[x \otimes t^m, y \otimes t^n] = \Big((m+1)xy - (n+1)yx\Big) \otimes t^{m+n}$$

where N is a left Novikov algebra, $x, y \in N$, $m, n \in \mathbb{Z}$.

Particular cases

- "Poisson brackets of hydrodynamic type" (I. Gelfand & I. Dorfman (1979–1981), A. Balinskii & S.P. Novikov (1985)).
- Schrödinger–Virasoro, Heisenberg–Virasoro (Y. Pei & C. Bai (2010–2012)).
- Finite-dimensional simple Lie algebras over $p = 2, 3$ **?**

(Co)homology of Lie algebras coming from dual operads

Question

Express (co)homology and other invariants (symmetric invariant bilinear forms, etc.) of $A \otimes B$ in terms of invariants of A and B.

Potential applications

- "Physics" (central extensions, 2-Lie algebras from (higher) gauge theory, ...)
- Structure theory of finite-dimensional Lie algebras in small characteristics.
- New invariants (cyclic cohomology-like?) of nonassociative algebras.

That's all. Thank you.

Slides at `http://justpasha.org/math/ihes.pdf`