Mock-Lie algebras

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Mock-Lie algebras

By definition, mock-Lie algebras are “commutative” Lie algebras:

$$xy = yx$$
$$(xy)z + (zx)y + (yz)x = 0$$

Fact 1

Mock-Lie algebras are Jordan algebras.

Proof

Setting $z = x$ in the Jacobi identity gives $x^2y = -2x(xy)$.
Replacing x by xy gives the Jordan identity $x^2(xy) = (x^2y)x$.

Fact 2

Mock-Lie algebras satisfy $x^3 = 0$.

Proof

Set $x = y = z$ in the Jacobi identity.

(All this assuming the characteristic of the base field $\neq 2, 3$).

Why bother?

- Certain genetic algebras are mock-Lie;
- Part of structure theory of Jordan algebras;
- Appear in classification of certain cyclic operads;
- It is curious to see what breaks in Lie algebras when anticommutativity is replaced by commutativity;
- **Possibility of realization of physically-motivated Lie algebras**;
- **Possible relation to QYBE**.

The dual life of mock-Lie algebras

nil-index 3 Jordan algebras ☯ strange cousins of Lie algebras

Fact 3

Finite-dimensional mock-Lie algebras are nilpotent.

Proof

Follows from the Jordan theory.

Some history

- 1960s-1970s Zhevlakov, Shestakov, et al.:
  "Jordan algebras of nil-index 3"
- 1976 Abraham: some kind of genetic algebras
- 1995 Getzler–Kapranov: "Mock-Lie algebras"
- 1997 Okubo–Kamiya: "Jordan-Lie algebras"
- 1997 Hentzel–Jacobs–Sverchkov
- 2014 Burde–Fialowski "Jacobi-Jordan algebras"
- 2015 Agore–Militaru "Jacobi-Jordan algebras"
- 2016 Gutierrez Fernandez–Garcia
- 2017 Zusmanovich

Examples

Example 1

"Heisenberg" algebra:

$$\langle x_1, \ldots, x_n, y_1, \ldots, y_n, z \rangle$$
$$x_i \cdot y_i = z$$

Example 2

Let V be a vector space.

$$V \oplus \bigwedge^{\varepsilon}(V)$$
$$(v, x) \cdot (w, y) = x \wedge w + y \wedge v, \quad \text{where } v, w \in V,\ x, y \in \bigwedge^{\varepsilon}(V)$$

Representations

(An old) Eilenberg's approach: consider $L \oplus V$ with square-zero V and an action of L on V.

$$\rho : L \to \mathrm{End}(V)$$
$$\rho(xy)(v) = -\rho(x)\rho(y)v - \rho(y)\rho(x)v, \quad \text{where } x, y \in L,\ v \in V$$

In particular, the role of derivations is played by antiderivations:

$$D(xy) = -D(x)y - xD(y)$$

Cohomology

Fact 4 (Getzler–Kapranov)

The mock-Lie operad is not Koszul.

Proof

The Koszul dual operad to mock-Lie is defined by

$$xy = -yx$$
$$(xy)z = -x(yz)$$

Apply the Ginzburg–Kapranov criterion:
$g_{\text{mock-Lie}}(g_{\text{dual mock-Lie}})(t) \neq t$

Question 1

Cohomology theory for mock-Lie algebras?

The standard ways to define low-degree cohomology – via *anti*derivations, central extensions, deformations – do not agree!

"Dual mock-Lie"

Question 2

Which "interesting" Lie algebras can be represented in the form $(A \otimes B)^{(-)}$, where A is a mock-Lie algebra, and B is an anticommutative antiassociative algebra?

Relation to QYBE

Fact 5 (Agore–Militaru)

Let A be a mock-Lie algebra, $z \in \mathrm{Z}(A)$, $\lambda \neq 0$ an element of the ground field. Then the map $A \otimes A \to A \otimes A$ defined by

$$a \otimes b \mapsto b \otimes a + \lambda z \otimes ab$$

is a solution of QYBE iff A satisfies $(xy)z = 0$.

Question 3

Generalization to arbitrary mock-Lie algebras? To other kinds of *YBE (QYBE with parameters, etc.)?

Special and exceptional algebras

Fact 6 (Hentzel–Jacobs–Sverchkov)

Mock-Lie algebras do not necessarily admit a faithful representation.

Equivalently:

- in Lie parlance: the Ado theorem fails for mock-Lie;
- in Jordan parlance: there are exceptional mock-Lie algebras, i.e. not every mock-Lie algebra can be embedded into an associative one (with respect to anticommutator $a \circ b = \frac{1}{2}(ab + ba)$).

Question 4

Which ones do? Or: which mock-Lie algebras are special and which are exceptional?

Special algebras

A (very) partial answer to Question 4

Mock-Lie algebras of dimension ≤ 6 are special.

1st proof

Using Gröbner base computations in GAP/GBNP.

2nd proof

Using direct computations with the kernel of the canonical map $L \to U(L)$.

3rd proof

Using some Jordan theory.

How to construct exceptional mock-Lie algebras

For any of the special Jordan identities (Glennie, Shestakov, Medvedev, ...), construct a mock-Lie algebra not satisfying this identity using Albert (`http://www1.osu.cz/~zusmanovich/albert/`).

A rigorous check that the given algebra does not satisfy the given identity in any characteristic $\neq 2, 3$ can be done in GAP.

The minimal dimension of so constructed algebra is 44.

Question 5 (a variation of Question 4)

What is:
1) the minimal dimension
2) the minimal degree of nilpotency
of an exceptional mock-Lie algebra?

A (very) partial answer

1) between 7 and 44.
2) between 6 and 9.

That's all. Thank you.

Slides at `http://www1.osu.cz/~zusmanovich/math.html`