Cohomology of tensor products

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A Lie-algebraic zoo. I

Current Lie algebras

$C^\infty(M, \mathfrak{g}) \simeq \mathfrak{g} \otimes C^\infty(M, \mathbb{R})$

Generalization

$L \otimes A$

L is a Lie algebra, A an associative commutative algebra

$[x \otimes a, y \otimes b] = [x, y] \otimes ab$

Closely related

Kac–Moody algebras

A Lie-algebraic zoo. II

"Poisson brackets of hydrodynamic type"
(S.P. Novikov et al.)

$$[u_i(x), u_j(y)] = g_{ij}(u(x))\delta'(x-y) + \sum_{k=1}^{n} \frac{\partial u_k}{\partial x} b_k^{ij}(u(x))\delta(x-y)$$

A Lie-algebraic zoo. III

Heisenberg–Virasoro, Schrödinger–Virasoro, etc.

Certain extensions/generalizations of the Witt algebra

"Affinization" of a Novikov algebra (Pei & Bai, 2010–2011)

$N \otimes \mathbb{C}[t, t^{-1}]$

$[x \otimes t^n, y \otimes t^m] = ((n+1)xy - (m+1)yx) \otimes t^{n+m}$

Generalization

$A \otimes B$

A is a left Novikov algebra, B a right Novikov algebra

$[a \otimes b, a' \otimes b'] = aa' \otimes bb' - a'a \otimes b'b$

Even more generalization

A, B are algebras over Koszul dual operads

A Lie-algebraic zoo. IV

“Lie algebras generated by dynamical systems” (Vershik et al.)

$C(X, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$
(X, T) is a dynamical system
$[f \otimes t^n, g \otimes t^m] = (f \cdot (g \circ T^n) - g \cdot (f \circ T^m)) \otimes t^{n+m}$

Generalization

$A \otimes B$
A, B are associative commutative algebras
generators of B act on A by automorphisms
$[a \otimes b, a' \otimes b'] = (a(a')^b - a'(a)^{b'}) \otimes bb'$

A Lie-algebraic zoo. V

(Some) Lie algebras of symmetries of differential equations

Symmetries of Khokhlov–Zabolotskaya, Boyer–Finley, etc., equations (O. Morozov, 2015)

$C^\infty(\mathbb{R},\mathbb{R}) \otimes \mathbb{R}[t]/(t^n) +$ tail of derivations

$[f \otimes t^i, g \otimes t^j] = \left(\sum_{k=0}^{i+j} \lambda_{ijk} f^{(k)} g^{(i+j-k)} \right) \otimes t^{i+j}$

A Lie-algebraic zoo

... and (much) more

Question

What these algebras have in common?

Answer

$A \otimes B$ with a "twisted" multiplication.

Now we have a zoo. What we can do with it?

Question

To compute (co)homology and other invariants of such algebras.

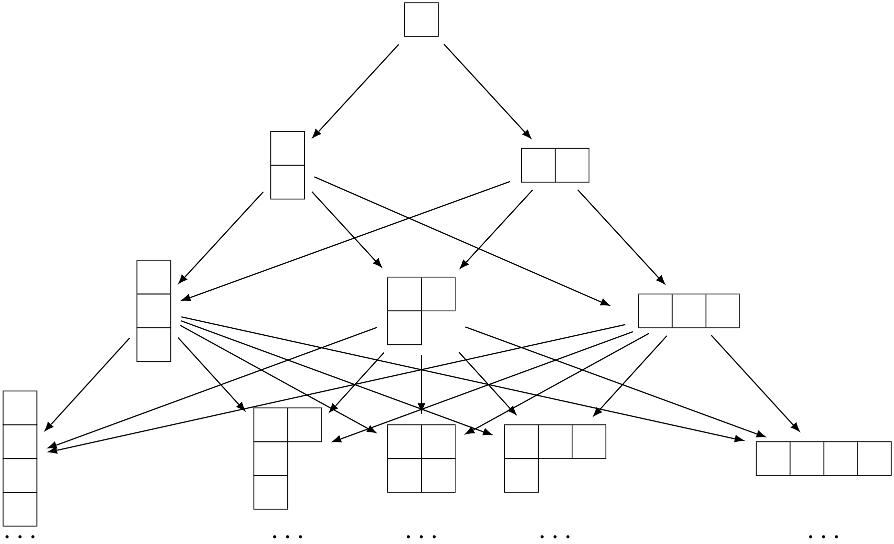
An attempt of answer

Decompose the Chevalley–Eilenberg complex according to the Cauchy formula:

$$\bigwedge^n (A \otimes B) \simeq \bigoplus_{\lambda} \mathsf{Y}_\lambda(A) \otimes \mathsf{Y}_{\lambda^\sim}(B)$$

Y_λ = Young symmetrizer corresponding to the Young diagram λ

How Young symmetrizers interact with the differential?



each Young diagram λ represents $Hom(Y_\lambda(L), ?) \otimes Hom(Y_\lambda(A), ?)$

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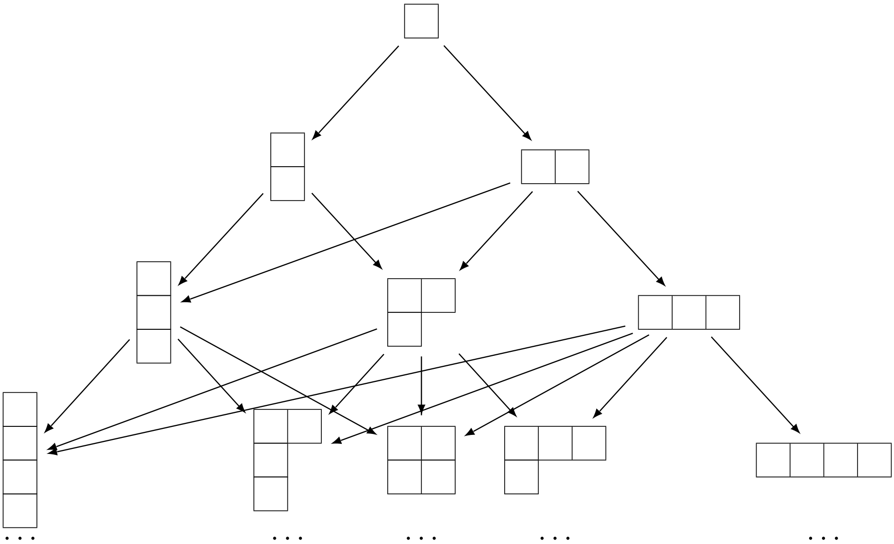

each Young diagram λ represents $Hom(Y_\lambda(L), ?) \otimes Hom(Y_\lambda(A), ?)$

A plethora of questions. I

Question

Why this miracle happens only for current Lie algebras?

Question

When some partial miracle may happen?

A (very) partial answer

- H^2 for "$A \otimes B$ over Koszul dual operads" when "noncommutative 2-cocycles" on A and B are "small" (captures the Heisenberg– and Schrödinger–Virasoro cases).
- H^2 for "Lie algebras generated by dynamical systems".
- H^2 for symmetry Lie algebras of some differential equations.

A plethora of questions. II

Algebras currently not in "the zoo" which I would like very much to include:

- Lax operator algebras (Krichever, Schlichenmaier, Sheinman)
- The Gaudin algebras $[S_i(x), S_j(y)] = \sum_k c_{ij}^k \frac{S_k(x) - S_k(y)}{x - y}$
- The Onsager algebra
- "Lie algebras over noncommutative rings" (Berenstein and Retakh)

That's all. Thank you.