Robinson–Amitsur ultrafilters, Jónsson's lemma, varieties of algebras, and Tarski's monsters

Pasha Zusmanovich

Tallinn University of Technology

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A theorem from 1960s

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Reminder

An (associative) ring R is called *prime* if one of the following equivalent conditions holds:

(i) $\forall\, I, J \triangleleft R \;\; I, J \neq 0 \Rightarrow IJ \neq 0$;

(ii) $\forall a, b \in R, a, b \neq 0 \; \exists x \in R : axb \neq 0$.

The proof

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An *ultrafilter* $\mathscr{U}$ on a set $\mathbb{I}$ is a set of subsets of $\mathbb{I}$ satisfying the following conditions:

1. $\varnothing \notin \mathscr{U}$
2. $\mathbb{X}, \mathbb{Y} \in \mathscr{U} \Rightarrow \mathbb{X} \cap \mathbb{Y} \in \mathscr{U}$
3. $\mathbb{X} \in \mathscr{U}, \mathbb{X} \subset \mathbb{Y} \Rightarrow \mathbb{Y} \in \mathscr{U}$
4. $\forall \mathbb{X} \subset \mathbb{I} \quad (\mathbb{X} \in \mathscr{U}) \vee (\mathbb{I} \backslash \mathbb{X} \in \mathscr{U})$ (maximality)

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The proof (continued)

Basic fact about ultrafilters

Any set $\mathcal{S}$ satisfying:

$$\mathbb{X}, \mathbb{Y} \in \mathcal{S} \Rightarrow \mathbb{X} \cap \mathbb{Y} \neq \varnothing$$

(finite intersection property)

contained in some filter and hence (by Zorn's lemma) in some ultrafilter.

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Reminder

Let $\mathscr{U}$ be an ultrafilter on $\mathbb{I}$. An *ultraproduct* of a set of rings $\{A_i\}_{i \in \mathbb{I}}$ is the ring

$$\prod_{\mathscr{U}} A_i = \Big(\prod_{i \in \mathbb{I}} A_i \Big) \Big/ \{ f \mid \{ i \in \mathbb{I} \mid f_i = 0 \} \in \mathscr{U} \}.$$

If $A_i \simeq A$, we have an *ultrapower* $A^{\mathscr{U}}$.

Łoś' theorem

For any first-order sentence φ,

$$\prod_{\mathscr{U}} A_i \models \varphi \quad \Leftrightarrow \quad \{i \in \mathbb{I} \mid A_i \models \varphi\} \in \mathscr{U}\,.$$

In particular,

$$A^{\mathscr{U}} \equiv A.$$

The proof (continued)

Define $\mathcal{S} = \{\{i \in \mathbb{I} \mid f_i \neq 0\} \mid f \in R, f \neq 0\}$.
$\mathcal{S}$ is **not** ultrafilter, but:

1. $\varnothing \notin \mathcal{S}$
2. ~~$\mathbb{X}, \mathbb{Y} \in \mathcal{S} \Rightarrow \mathbb{X} \cap \mathbb{Y} \in \mathcal{S}$~~
   $\mathbb{X}, \mathbb{Y} \in \mathcal{S} \Rightarrow \exists \mathbb{Z} \in \mathcal{S} : \mathbb{Z} \subset \mathbb{X} \cap \mathbb{Y}$

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Proof of 2.
$\mathbb{X} = \{i \in \mathbb{I} \mid f_i \neq 0\},\ 0 \neq f \in R$
$\mathbb{Y} = \{i \in \mathbb{I} \mid g_i \neq 0\}, 0 \neq g \in R$
Since R is prime, there is $x \in R$ such that $fxg \neq 0$, and

$$\begin{aligned}\mathbb{Z} = \{i \in \mathbb{I} \mid (fxg)_i = f_i x_i g_i \neq 0\} &\subset \{i \in \mathbb{I} \mid f_i \neq 0\} = \mathbb{X} \\ &\subset \{i \in \mathbb{I} \mid g_i \neq 0\} = \mathbb{Y}.\end{aligned}$$

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$\mathcal{S}$ satisfies the finite intersection property and **is contained in** some ultrafilter $\mathscr{U}$.

$$R = R \Big/ \Big(R \cap \{ f \in \prod_{i \in \mathbb{I}} A_i \mid \{ i \in \mathbb{I} \mid f_i = 0 \} \in \mathscr{U} \} \Big) \subseteq \prod_{\mathscr{U}} A_i .$$

Q.E.D.

A generalization

Theorem ("Robinson–Amitsur for algebraic systems")

Let $\{B_i\}_{i\in\mathbb{I}}$ be a set of algebraic systems from an ideal-determined class, A a finitely directly irreducible algebraic system. Then

$$A \subseteq \prod_{i\in\mathbb{I}} B_i \;\Rightarrow\; \exists \text{ ultrafilter } \mathscr{U} \text{ on } \mathbb{I} : A \subseteq \prod_{\mathscr{U}} B_i.$$

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Reminder 1

A class of algebraic systems is *ideal-determined* if its congruences behave "good enough" – can be described in terms of ideals.

Examples: rings, groups, Ω-groups (P.J. Higgins, Kurosh).

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Reminder 2

An algebraic system A (from an ideal-determined class) is *finitely subdirectly irreducible* if one of the following equivalent conditions holds:

(i) $\forall i \in \mathbb{I} \quad 0 \neq I_i \lhd A,\ |\,\mathbb{I}\,| < \aleph_0 \;\Rightarrow\; \bigcap_{i\in\mathbb{I}} I_i \neq 0;$

(ii) $A \subseteq \prod_{i\in\mathbb{I}} B_i,\ \ |\,\mathbb{I}\,| < \aleph_0 \;\Rightarrow\; \exists i \in \mathbb{I} : A \subseteq B_i.$

For rings and algebras, primeness $\Rightarrow$ finite subdirect irreducibility.

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(ii) $A \subseteq \prod_{i\in\mathbb{I}} B_i$, ~~$|\mathbb{I}| < \aleph_0$~~ $\Rightarrow \exists i \in \mathbb{I} : A \subseteq B_i$.

Monolith of $A = \bigcap_{0\neq I\triangleleft A} I$.

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(Generalized) Jónsson's lemma (Freese, Hagemann, Herrmann, Hrushovski, McKenzie)

Let $\{B_i\}_{i\in\mathbb{I}}$ be a set of algebraic systems from a modular variety (i.e., the lattice of congruences is modular), A a subdirectly irreducible algebraic system. Then

$$\begin{aligned} A \in Var(\{B_i\}_{i\in\mathbb{I}}) \Rightarrow \exists \text{ ultrafilter } \mathscr{U} \text{ on } \mathbb{I} :\\ A/(\text{centralizer of the monolith of } A) \subseteq\\ \text{homomorphic image of a subsystem of } \prod_{\mathscr{U}} B_i . \end{aligned}$$

Birkhoff meets Robinson–Amitsur

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Corollary (Birkhoff + Robinson–Amitsur)

A finitely subdirectly irreducible free algebra $\mathcal{F}$ in an (ideal-determined) variety $Var(A)$ or quasivariety $Qvar(A)$ embeds in an ultrapower $A^{\mathscr{U}}$.

Proof. By Birkhoff's theorem, $\mathcal{F} = B/I$ for some $B \subseteq A^{\mathbb{I}}$ and $I \triangleleft B$. By the universal property of $\mathcal{F}$, it embeds in B and hence in $A^{\mathbb{I}}$. Apply Robinson–Amitsur for algebraic systems.

Criterion for absence of non-trivial identities for algebraic systems

Let $\mathfrak{V}$ be a variety of algebraic systems from an ideal-determined class, and suppose that all free systems of $\mathfrak{V}$ are finitely subdirectly irreducible. Then for an algebraic system $A \in \mathfrak{V}$, the following are equivalent:

(i) any identity of A is an identity of $\mathfrak{V}$ (i.e., A does not satisfy nontrivial identities within $\mathfrak{V}$);

(ii) any free system of $\mathfrak{V}$ embeds in an ultrapower of A;

(iii) any free system of $\mathfrak{V}$ embeds in a system elementarily equivalent to A.

Proof.

(i) $\Rightarrow$ (ii) follows from "Birkhoff + Robinson–Amitsur".
(ii) $\Rightarrow$ (iii) follows from Łoś' theorem.
(iii) $\Rightarrow$ (i) is trivial.

Criterion for absence of non-trivial identities for algebras

For an algebra A over a field K belonging to one of the following varieties of algebras: all algebras, associative algebras, or Lie algebras, the following are equivalent:

(i) A does not satisfy a nontrivial identity;

(ii) any free algebra embeds in an ultrapower of A;

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Proof. Follows from the criterion for algebraic systems.

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Proof. Follows from the criterion for algebraic systems.
Two issues:

- Whether free algebras are finitely subdirectly irreducible?
- "Birkhoff + Robinson–Amitsur" gives embedding on the level of K-algebras, not of $K^{\mathscr{U}}$-algebras.

In which varieties free algebras are finitely subdirectly irreducible?

variety		**why?**
all algebras	yes	no zero divisors
associative	yes	
Lie	yes	Shirshov–Witt theorem
alternative	no	explicit example
Jordan	no	explicit example

(If the answer is "yes", they are even prime).

K-algebras vs. $K^{\mathscr{U}}$-algebras

"Birkhoff + Robinson–Amitsur" establishes embedding (under appropriate conditions) of K-algebras:

$$\mathcal{F} \subseteq A^{\mathscr{U}}.$$

To be able to apply Łoś' theorem, one needs embedding of $K^{\mathscr{U}}$-algebras:

$$K^{\mathscr{U}} \mathcal{F} \subseteq A^{\mathscr{U}}.$$

By the universal property of the tensor product, we have a surjection

$$\mathcal{F} \otimes_K K^{\mathscr{U}} \to K^{\mathscr{U}} \mathcal{F},$$

but, generally, this is far from being an isomorphism. This is so, however, if $\mathcal{F}$ does not have commutative subspaces of dimension > 1, in particular, for absolutely free, free associative and free Lie algebras.

Criterion for absence of non-trivial identities for groups

Theorem

For a group G belonging to one of the following varieties: all groups, groups satisfying the identity $x^p = 1$ for a prime $p > 665$, the following are equivalent:

(i) G does not satisfy a nontrivial identity (within the given variety);

(ii) any free group in the variety embeds in an ultrapower of G;

(iii) any free group in the variety embeds in a group elementary equivalent to G.

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Question

Semigroups?

Application: PI

Theorem (Regev, 1972)

If A, B are PI, then $A \otimes B$ is PI.

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If $A = M_n(K)$, and B is PI, then $A \otimes B$ is PI.

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Theorem

If A is finite-dimensional, B is PI, then $A \otimes B$ is PI.

Proof. Suppose the contrary. Then a free associative algebra of exponential growth embeds in

$$(A \otimes_K B)^{\mathscr{U}} \simeq (A \otimes_K K^{\mathscr{U}}) \otimes_{K^{\mathscr{U}}} B^{\mathscr{U}}$$

of polynomial growth, a contradiction.

Application: algebras with same identities

Theorem (Razmyslov, 1982)

If $\mathfrak{g}_1, \mathfrak{g}_2$ are finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 0, then

$$Var(\mathfrak{g}_1) = Var(\mathfrak{g}_2) \;\Leftrightarrow\; \mathfrak{g}_1 \simeq \mathfrak{g}_2.$$

Theorem (Drensky–Racine, 1992)

Ditto for finite-dimensional simple Jordan algebras.

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Ditto for finite-dimensional simple Jordan algebras.

Joint proof. Using the fact that in these classes of algebras primeness is equivalent to simplicity, and by the embedding machinery above, reduced to the case where $\mathfrak{g}_1 \subseteq \mathfrak{g}_2$, which follows easily from the known structural results.

Application: Tarski's monsters

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Growth sequence

$g_n(G)$ = number of generators of $G \times \cdots \times G$ (n times).
Wise (2002): groups with $g_n \equiv 2$.
Garion & Glasner (2010): for a Tarski's monster, $g_n \leq 3$.

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Theorem

(i) A Tarski's monster all whose proper subgroup are infinite cyclic does not satisfy a nontrivial identity iff it has infinite girth.

(ii) A Tarski's monster all whose proper subgroups are cyclic of order p does not satisfy a nontrivial identity except $x^p = 1$ and its consequences, iff it has infinite relative girth.

(iii) For a Tarski's monster satisfying condition (i) or (ii), $g_n \equiv 2$.

Speculations

Lie-algebraic analogs of Tarski's monsters

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Tits' alternative

Tits (1972): a linear group contains either a solvable subgroup of finite index, or a nonabelian free subgroup.
Platonov (1967): a linear group which satisfies a nontrivial identity, contains a solvable subgroup of finite index.

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Platonov (1967): a linear group which satisfies a nontrivial identity, contains a solvable subgroup of finite index.

Jacobson's problem

In a Lie p-algebra, does $x^{p^{n(x)}} = x$ imply abelianity?
Weaker question: does it imply a nontrivial identity?

That's all. Thank you.

Based on arXiv:0911.5414
Slides at http://justpasha.org/math/ultra/