Contact brackets and other structures on the tensor product

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The purpose of this report is once more to call attention to an elementary and, in some cases, very effective technique of computing various kinds of structures on tensor products. Such problems often can be reduced to the simultaneous evaluation of kernels of several tensor product maps, i.e., maps of the form $S \otimes T$, where S and T are linear operators on the respective spaces of linear maps. Using the fact that

$$(\bigstar) \qquad \mathrm{Ker}(S \otimes T) = \mathrm{Ker}(S) \otimes * + * \otimes \mathrm{Ker}(S),$$

the question reduces to evaluation of the intersection of several linear spaces having the form as on the right-hand side of ($\bigstar$), for various operators S and T. The intersection of two such spaces satisfies the distributivity, and so can be handled effectively, due to the following elementary linear algebraic lemma:

Lemma (Lemma 1.1 in [4]). *Let U_1, U_2 be subspaces of a vector space U, and V_1, V_2 be subspaces of a vector space V. Then*

$$(U_1 \otimes V + U \otimes V_1) \cap (U_2 \otimes V + U \otimes V_2) = (U_1 \cap U_2) \otimes V + U_1 \otimes V_2 + U_2 \otimes V_1 + U \otimes (V_1 \cap V_2).$$

This technique was used for the first time in [4] to derive some formulas for the low degree cohomology of current Lie algebras, i.e., Lie algebras of the form $L \otimes A$, where L is a Lie algebra, and A is an associative commutative algebra. The paper [5] contains further results about such cohomology, as well as about Poisson and Hom-Lie structures on current and related Lie algebras. The last our result in this direction is in [6], which answers a recent question from [2] about extension of contact bracket on the tensor product from the bracket on the factors.

Recall that the contact bracket on a commutative associative algebra A with unit is a bilinear map $[\cdot, \cdot] : A \times A \to A$ such that

$$[ab, c] = [a, c]b + [b, c]a + [c, 1]ab$$

for any $a, b, c \in A$. Contact brackets are an obvious generalization of Poisson brackets, the latter being contact brackets satisfying $[A, 1] = 0$. It was asked in [2] whether, given contact brackets on two algebras A and B, is it always possible to extend them to the tensor product $A \otimes B$? In [6], using some general formulas for the space of contact brackets on some particular classes of algebras, a procedure was devised for constructing examples showing that such extension is not always possible.

This linear algebraic method is sometimes very effective, but its applicability is severely limited by the fact that no statement similar to Lemma is true for intersection of three or more spaces. The proper contexts of Lemma might be criteria for distributivity of a set of subspaces of a vector space (for an exposition, see, for example, [3, Chap. 1, §7]) and, more speculatively, the "four subspaces problem" of Gelfand–Ponomarev, [1].

References

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