Octonion matrix algebras

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Octonion matrix algebras

$S^+(M_n(\mathbb{O}), J) = \{A \in M_n(\mathbb{O}) \mid J(A) = A\}$ Hermitian

$S^-(M_n(\mathbb{O}), J) = \{A \in M_n(\mathbb{O}) \mid J(A) = -A\}$ skew-Hermitian

$J : (a_{ij}) \mapsto (\overline{a_{ji}})$ an involution

Why bother?

$\boldsymbol{S}^{+}(\boldsymbol{M_n}(\mathbb{O}), \boldsymbol{J})$:

$n = 1$: the ground field K

$n = 2$: the 10-dimensional simple Jordan algebra of symmetric bilinear form

$n = 3$: the famous 27-dimensional exceptional simple Jordan algebra

$n \geq 4$: no longer Jordan, but appear in M-theory

$\boldsymbol{S}^{-}(\boldsymbol{M_n}(\mathbb{O}), \boldsymbol{J})$:

$n = 1$: the 7-dimensional simple Malcev algebra $\mathbb{O}^{-}$.

Simplicity

Theorem 1

The algebras $S^+(M_n(\mathbb{O}), J)$ and $S^-(M_n(\mathbb{O}), J)$ are simple.

Method of the proof: use realizations

$$S^{\pm}(M_n(\mathbb{O}), J) \simeq M_n^{\pm}(K) \otimes 1 + M_n^{\mp}(K) \otimes \mathbb{O}^-,$$

and a variant of the Jacobson density theorem.

δ-derivations and associative forms

$$D(xy) = \delta D(x)y + \delta x D(y)$$

Theorem 2

δ-derivations of $S^+(M_n(\mathbb{O}), J)$ and $S^-(M_n(\mathbb{O}), J)$ are trivial (i.e., either the usual derivations, or multiplications by a scalar).

(Earlier derivations were computed by H. Petyt).

Theorem 3

Symmetric associative forms on $S^+(M_n(\mathbb{O}), J)$ and $S^-(M_n(\mathbb{O}), J)$ are:

$$(X, Y) \mapsto \mathrm{Tr}\,(XY + \overline{X}\,\overline{Y}).$$

Further questions

1) Automorphisms? Conjecture: $G_2 \times SO(n)$.

2) Identities?

3) Subalgebras?

Some references

- M. Bremner and I. Hentzel, *Identities for algebras of matrices over the octonions*, J. Algebra **277** (2004), no.1, 73–95.
- P. Jordan, *Zur Theorie nicht-assoziativer Algebren*, Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl. 1968, no.2, 27–38.
- J. Lukierski and F. Toppan, *Generalized space-time supersymmetries, division algebras and octonionic M-theory*, Phys. Lett. B **539** (2002), no.3-4, 266–276.
- H. Petyt, *Derivations of octonion matrix algebras*, Comm. Algebra **47** (2019), no.10, 4216–4223.
- H. Rühaak, *Matrix-Algebren über einer nicht-ausgearteten Cayley-Algebra*, PhD Thesis, Univ. Hamburg, 1968.
- A. Zohrabi and P. Zusmanovich, *On Hermitian and skew-Hermitian matrix algebras over octonions*, J. Nonlin. Math. Phys., to appear.
- ______ and ______, *A δ-first Whitehead Lemma*, Manuscript.

That's all. Thank you.