Novikov structures on Kac-Moody and modular semisimple Lie algebras

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What is a left-symmetric structure?

Left-symmetric (aka Vinberg, pre-Lie, chronological) identity:

$$x(yz) - (xy)z = y(xz) - (yx)z$$

Equivalently:

- $(x, y, z) = (y, x, z)$, where $(x, y, z) = (xy)z - x(yz)$ is the associator.
- $[L_x, L_y] = L_{[x,y]}$, where $L_x(a) = xa$.

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Left-symmetricity $\Rightarrow$ Lie-admissibility:

$$[x, y] = xy - yx$$

satisfies the Jacobi identity.

Question

Describe left-symmetric structures on a given Lie algebra.

Origin

Theory of affine manifolds (Auslander, Milnor).

What is a Novikov structure?

Left-symmetric +

$$(xy)z = (xz)y$$

Equivalently:

$$[R_x, R_y] = 0$$

where $R_x(a) = ax$.

Question

Describe Novikov structures on a given Lie algebra.

Origin (of Novikov algebras)

- Integrability of dynamical systems (Gelfand & Dorfman).
- Poisson brackets of hydrodynamic type (Balinsky & Novikov).

What is known about Novikov structures?

($p = 0$, finite-dimensional, unless stated otherwise)

- Helmstetter (1979): a Lie algebra admitting a left-symmetric structure is not perfect ($[L, L] \neq L$).
- Osborn (1992) and Xu (1996): Novikov structures on the Zassenhaus algebra $W_1(n)$ ($p > 0$).
- Osborn & Zelmanov (1995): an inverse problem: Lie structures on some infinite-dimensional Novikov Witt-type algebras.
- Xu (2000): same inverse problem; Novikov structures on the infinite-dimensional Witt algebra.
- Bai & Meng (2001): Novikov structures on 4-dimensional nilpotent Lie algebras.
- Burde (2006): a Lie algebra admitting a Novikov structure is solvable.
- Burde, Dekimpe & Vercammen (2008): Novikov structures on Lie algebras of (strictly) upper triangular $n \times n$ matrices exist only for small n.

Quadratic vs. linear problems

Suppose a Lie-admissible algebra satisfies a number of identities:

$$\sum_{\sigma\in S_3} \Big(\alpha^i_\sigma (x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)} + \beta^i_\sigma x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)})\Big) = 0, \quad i = 1, \dots, n$$

Rewrite these identities in terms of $[\,\cdot\,,\cdot\,]$ and $\circ$, where

$$xy = \frac{1}{2}[x,y] + \frac{1}{2}x\circ y \qquad (\text{so } x\circ y = xy + yx):$$

$$\begin{aligned}
&(\alpha^i_e + \alpha^i_{(12)} + \beta^i_e + \beta^i_{(12)})(x_1\circ x_2)\circ x_3 \\
+&(\alpha^i_{(23)} + \alpha^i_{(132)} + \beta^i_{(23)} + \beta^i_{(132)})(x_3\circ x_1)\circ x_2 \\
+&(\alpha^i_{(13)} + \alpha^i_{(123)} + \beta^i_{(13)} + \beta^i_{(123)})(x_2\circ x_3)\circ x_1 \\
+&(\text{terms linear with respect to } \circ) = 0.
\end{aligned}$$

Quadratic problem

For a given Lie algebra $(L, [\,\cdot\,,\cdot\,])$, find symmetric maps $L\circ L \to L$, satisfying these identities.

Quadratic vs. linear problems (continuation)

If

$$rk \begin{pmatrix} \alpha_e^1 + \alpha_{(12)}^1 + \beta_e^1 + \beta_{(12)}^1 & \alpha_{(23)}^1 + \alpha_{(132)}^1 + \beta_{(23)}^1 + \beta_{(132)}^1 & \alpha_{(13)}^1 + \alpha_{(123)}^1 + \beta_{(13)}^1 + \beta_{(123)}^1 \\ \dots & \dots & \dots \\ \alpha_e^n + \alpha_{(12)}^n + \beta_e^n + \beta_{(12)}^n & \alpha_{(23)}^n + \alpha_{(132)}^n + \beta_{(23)}^n + \beta_{(132)}^n & \alpha_{(13)}^n + \alpha_{(123)}^n + \beta_{(13)}^n + \beta_{(123)}^n \end{pmatrix} < n$$

then this quadratic problem has a **linear** consequence.

Quadratic vs. linear problems (continuation)

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Left-symmetric structures - quadratic

Novikov (and LR, and probably some others?) structures - linear!

2-sided Alia algebras

(Dzhumadil'daev, 2009)

$$[x, y]z + [z, x]y + [y, z]x = 0$$
$$z[x, y] + y[z, x] + x[y, z] = 0$$

commutative
Lie
Novikov
LR

$\Rightarrow$ 2-sided Alia $\Rightarrow$ Lie-admissible

Question

Describe 2-sided Alia structures on a given Lie algebra.

2-sided Alia algebras

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Question

Describe 2-sided Alia structures on a given Lie algebra.

Equivalent question

Describe commutative 2-cocycles on a given Lie algebra.

What are commutative 2-cocycles?

Each 2-sided Alia structure on a Lie algebra L is given by

$$xy = [x, y] + \varphi(x, y)$$

where $\varphi : L \times L \to K$ is a **commutative** 2-**cocycle** on L, i.e.:

1. φ is symmetric
2. $\varphi([x, y], z) + \varphi([z, x], y) + \varphi([y, z], x) = 0$

Cocycles with $\varphi([L, L], L) = 0$ are called **trivial**.

The space of all K-valued commutative 2-cocycles on L is denoted as $Z^2_{comm}(L)$.

Commutative 2-cocycles on current Lie algebras

A **current Lie algebra** is a Lie algebra of the form $L \otimes A$, where L is a Lie algebra, A is an associative commutative algebra, with the Lie bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

for $x, y \in L$, $a, b \in A$.

Fact

Under some technical assumptions,

$$\begin{aligned} & Z^2_{comm}(L \otimes A) \\ & \simeq Z^2_{comm}(L) \otimes A^* \\ & \oplus \{\varphi : L \times L \to K \mid \varphi \text{ is skew; } \varphi([x, y], z) = \varphi([z, x], y)\} \otimes HC^1(A) \\ & \oplus \text{(trivial cocycles)} \end{aligned}$$

No Novikov structures on Kac-Moody

Kac–Moody:

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}t\frac{d}{dt} + \mathbb{C}z$$

Corollary

All commutative 2-cocycles on affine Kac-Moody algebras are trivial.

Fact

A Lie algebra which is not 2-step solvable and all whose commutative 2-cocycles are trivial, do not admit a Novikov structure.

Theorem

Affine Kac–Moody algebras do not admit Novikov structures.

Novikov structures on modular semisimples

Modular semisimple Lie algebras:

$$S \otimes K[t_1, \ldots, t_n]/(t_1^p, ..., t_n^p) + 1 \otimes D$$

Another corollary

Z^2_{comm} of such algebras is isomorphic to $Z^2_{comm}(D)$.

Another theorem

A modular semisimple Lie algebra admits a Novikov structure if and only if $W_1(n)$ is “involved”.

Novikov structures on modular semisimples

Modular semisimple Lie algebras:

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Another corollary

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Another theorem

A modular semisimple Lie algebra admits a Novikov structure if and only if $W_1(n)$ is “involved”.

Simples:

- Burde (1994): left-symmetric structures on classical, examples for Cartan type.
- Dzhumadil’daev & Zusmanovich (2010): nonzero commutative 2-cocycles exist only on $sl(2)$ and $W_1(n)$.

Semisimples: by the corollary above.

That's all. Thank you.

Slides at http://justpasha.org/math/oostende.pdf