Structure functions and Spencer cohomology in zero and positive characteristics

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Structure functions

Let $G \subseteq GL(n)$ be a real of complex Lie group, M an n-dimensional real of complex manifold.

A *G-structure* on M is a reduction of the principal $GL(n)$-bundle to the principal G-bundle.

Structure functions are obstructions to integrability (= local flattening) of M endowed with a G-structure).

Some (well known) particular cases:

G	**name of a G-structure**	**name of a structure function**
$O(n)$	Riemann metric	Riemann tensor
$O(n) \times \mathbb{R}^*$	almost conformal structure	Weyl tensor
$GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$	almost complex structure	Nijenhuis tensor

Spencer cohomology

Structure functions are interpreted in terms of the *Spencer cohomology* $H^*(\mathscr{L}_{-1}, \mathscr{L})$ of a graded Lie algebra $\mathscr{L} = \bigoplus_{n \geq -1} \mathscr{L}_n$.

Major examples of $\mathscr{L}$: Lie algebras of Cartan type W_n, S_n, H_{2n}.

Theorem (Serre)

The Spencer cohomology vanishes in degrees > 0 for W_n and S_n, and is fully computed for H_{2n}.

Spencer cohomology is also responsible for *filtered deformations* of a graded Lie algebra $\mathscr{L}$, and therefore important for characteristic $p > 0$ analogs of Lie algebras of Cartan type.

The algebras $\mathbb{W}(L, A)$

Let L be an abelian Lie algebra acting by derivations on an associative commutative algebra A, such that AL is a free submodule of $Der(A)$.

The Lie algebra $\mathbb{W}(L, A)$ is defined as the vector space $AL \simeq L \otimes A$ with multiplication

$$[x \otimes a, y \otimes b] = y \otimes ax(b) - x \otimes by(a).$$

The algebras $\mathbb{W}(L, A)$ (cont.)

Particular cases of the construction from the previous slide are:

1. $A = K[t_1, \dots, t_n]$, $L = \langle \frac{\mathrm{d}}{\mathrm{d}\, t_1}, \dots, \frac{\mathrm{d}}{\mathrm{d}\, t_n} \rangle$:
   $\mathbb{W}(L, A)$ = one-sided Jacobson–Witt algebra =
   infinite-dimensional Lie algebra of the general Cartan type W_n
   = Lie algebra of polynomial vector fields on the plane K^n.
2. $A = K[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, $L = \langle \frac{\mathrm{d}}{\mathrm{d}\, t_1}, \dots, \frac{\mathrm{d}}{\mathrm{d}\, t_n} \rangle$:
   $\mathbb{W}(L, A)$ = two-sided Jacobson–Witt algebra = Lie algebra of polynomial vector fields on the n-dimensional sphere.
3. K is of characteristic $p > 0$, $A = O(n; \overline{m})$, the algebra of divided powers in n variables with shearing parameters $\overline{m} = (m_1, \dots, m_n)$, $L = \langle \partial_1, \dots, \partial_n \rangle$:
   $\mathbb{W}(L, A)$ = finite-dimensional Lie algebra of the general Cartan type $W(n; \overline{m})$.

A unified approach to calculation of the Spencer cohomology: case W_n

Theorem

Let L has a basis $D_1, \ldots, D_n$ such that the algebra A decomposes as the tensor product of algebras $A_1 \otimes \cdots \otimes A_n$, with D_i acting on A_i. Then

$$\mathrm{H}^k(L, A) \simeq \bigoplus_{1 \le i_1 < \cdots < i_k \le n} A_1^{D_1} \otimes \cdots \otimes (A_{i_1})_{D_{i_1}} \otimes \cdots \otimes (A_{i_k})_{D_{i_k}} \otimes \cdots \otimes A_n^{D_n}.$$

Sketch of the proof

1) $H^k(L, \mathbb{W}(L, A)) \simeq L \otimes H^k(L, A)$.
2) Apply the Künneth formula.

Corollaries

Serre's vanishing result in $p = 0$, and non-vanishing result in $p > 0$.

A unified approach to calculation of the Spencer cohomology: case S_n

The Lie algebra $\mathbb{S}(L, A)$ is defined as the kernel of homomorphism

$$div : \mathbb{W}(L, A) \to A, \quad x \otimes a \to x(a).$$

To compute the corresponding Spencer cohomology, apply the cohomology long exact sequence associated with the short exact sequence of L-modules

$$0 \to \mathbb{S}(L, A) \to \mathbb{W}(L, A) \stackrel{div}{\to} L(A) \to 0.$$

A unified approach to calculation of the Spencer cohomology: case H_{2n}

Let $(D_1, \dots, D_n)$ and $(F_1, \dots, F_n)$ be two n-element sets of pairwise commuting derivations of A. Then A equipped with the bracket

$$[a, b] = \sum_{i=1}^{n} \Big(D_i(a)F_i(b) - F_i(a)D_i(b) \Big)$$

is a generalization of all kinds of Hamiltonian Lie algebras.

To compute the corresponding Spencer cohomology, apply considerations based on the Künneth formula, similar to the case of $\mathbb{W}(L, A)$ (but more cumbersome).

That's all. Thank you.