

Commutative cohomology in characteristic 2

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Commutative Lie algebras

Standing assumption: $p = 2$

$$[x, y] = [y, x]$$

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$$

$$\text{Lie} \subset \text{commutative Lie} \subset \text{Leibniz}$$

$$[x, x] = 0$$

Commutative cohomology

$$0 \rightarrow S^0(L, M) \xrightarrow{d} S^1(L, M) \xrightarrow{d} S^2(L, M) \xrightarrow{d} \dots$$

$S^n(L, M)$ = the space of n -linear *symmetric* maps $L \times \dots \times L \rightarrow M$

$$\begin{aligned} d\varphi(x_1, \dots, x_{n+1}) &= \sum_{1 \leq i < j \leq n+1} \varphi([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{n+1}) \\ &+ \sum_{i=1}^{n+1} x_i \bullet \varphi(x_1, \dots, \widehat{x_i}, \dots, x_{n+1}). \end{aligned}$$

Why bother?

- 1) Operadic viewpoint: algebras should be defined by multilinear identities.
- 2) New phenomena in cohomology, similar to Lie superalgebras.
- 3) New invariant of (ordinary) Lie algebras.
- 4) Appears naturally in the context of classification of simple Lie algebras.

Relation to Chevalley–Eilenberg and Leibniz cohomology

$$H^*(L, M) \rightarrow H_{comm}^*(L, M) \rightarrow HL^*(L, M)$$

Neither of these maps is an injection or a surjection.

Cup product

$$(\varphi \smile \psi)(x_1, \dots, x_{p+q}) = \sum_{\text{shuffles}} \varphi(x_{i_1}, \dots, x_{i_p}) \cdot \psi(x_{j_1}, \dots, x_{j_q})$$

$\smile$ turns $H_{comm}^*(L, K)$ into an associative graded ring.

Some computations

Adding up SIMILAR TERMS

$$-\sum_{k=2}^{n+1} \delta_{x_k} F_c(x_1, \dots, \hat{x}_k, \dots, x_{n+1}) + \sum_{k=2}^{n+1} \delta_{x_k} F_c(x_1, \dots, \hat{x}_k, \dots, x_{n+1}) + \sum_{1 \leq p < q \leq n+1} \delta_{x_p} \delta_{x_q} F_a(a, x_2, \dots, \hat{x}_p, \dots, \hat{x}_q, \dots, x_{n+1}) - \sum_{1 \leq p < q \leq n+1} \delta_{x_p} \delta_{x_q} F_a(a, x_2, \dots, \hat{x}_p, \dots, \hat{x}_q, \dots, x_{n+1}) = 0,$$

$$\sum_{1 \leq p < q \leq n+1} \delta_{x_p} \delta_{x_q} F_c(a, x_2, \dots, \hat{x}_p, \dots, \hat{x}_q, \dots, x_{n+1}) - \sum_{1 \leq p < q \leq n+1} \delta_{x_p} \delta_{x_q} F_c(a, x_2, \dots, \hat{x}_p, \dots, \hat{x}_q, \dots, x_{n+1}) = 0,$$

$$\sum_{1 \leq p < q \leq n+1} \delta_{x_p} \delta_{x_q} F_c(a, x_2, \dots, \hat{x}_p, \dots, \hat{x}_q, \dots, x_{n+1}) - \sum_{1 \leq p < q \leq n+1} \delta_{x_p} \delta_{x_q} F_c(a, x_2, \dots, \hat{x}_p, \dots, \hat{x}_q, \dots, x_{n+1}) = 0.$$

1) $F(a, a, \dots, a)$ is NOT CONTAINED IN $\mathcal{P}$

$$2) F(a, \dots, a) \neq 0 \Rightarrow F_c(a, \dots, a) + F_c(a, \dots, a) + \dots + F_c(a, \dots, a) = 0 \Rightarrow F_c(a, \dots, a) = 0 \text{ if } n+1 \equiv 0 \pmod{2}$$

$$3) \text{ SIMILARLY ONE CAN OBTAIN } F_c(a, \dots, a) = 0 \text{ if } n+1 \equiv 0 \pmod{2}$$

$$4) \left(\frac{a, a, \dots, a, c, a, \dots, a}{s_1, t_1+2=n+1 \Rightarrow s_1+t_1=n+1} \right) \quad 5) \left(\frac{a, \dots, a, b, \dots, a, c, a, \dots, a}{s_1, t_1+1=n+1 \Rightarrow s_1+t_1=n+1} \right) \text{ not in } \mathcal{P}$$

$$1): F_c(a, a, \dots, a, a) + F_c(a, a, \dots, a, c, a) + F_c(a, a, \dots, a, a, a) = 0$$

$$(2): F_c(a, a, \dots, a, a) = 0$$

$$(3): F_c(a, a, \dots, a, a) = 0$$

$$(1) F_c(a, a, \dots, a, b, a, a) + F_c(a, a, \dots, a, c, a, a) = 0$$

$$(2) F_c(a, a, \dots, a, a, a, a) = 0$$

$$(3) F_c(a, a, \dots, a, a, a, a) = 0$$

SINCE $F(x_1, \dots, x_{n+1}) = F(x_1, \dots, x_n) \forall x_{n+1} \in \mathbb{C}$

then we can consider the following non equivalent 2-tuples

$$x_{r,p,q} = (a, \dots, a, b, \dots, b, c, \dots, c) = (r, p, q)$$

$$x_{r',p',q'} = x_{r,p,q} \text{ iff } r=r', p=p', q=q'$$

Thus we get

$$F_a(r+1, p-1, q-1) + F_c(r, p-1, q) + F_c(r, p, q-1) = 0$$

$$\varphi(x) = F_a(x) + F_c(x) + F_c(x) \text{ IS A COCYCLE.}$$

2-dimensional nonabelian

$$L = \langle a, b \mid [a, b] = a \rangle$$

$$\chi_{pq}(\underbrace{a, \dots, a}_r, \underbrace{b, \dots, b}_s) = \begin{cases} 1 & \text{if } p = r \text{ and } q = s \\ 0 & \text{otherwise} \end{cases}$$

$$H_{comm}^n(L, K) \simeq \langle \chi_{pq} \mid p + q = n, p \text{ even} \rangle$$

$$\chi_{pq} \smile \chi_{rs} = \binom{p+r}{p} \binom{q+s}{q} \chi_{p+r, q+s}$$

Zassenhaus

$$W'_1(n) = \langle e_i = x^{(i+1)}\partial \mid -1 \leq i \leq 2^n - 3 \rangle$$

$$\dim H^2_{comm}(W'_1(n), K) = n$$

Basic cocycles:

$$e_i \vee e_j \mapsto \begin{cases} 1 & \text{if } i = j = 2^k - 2, \text{ or } \{i, j\} = \{-1, 2^{k+1} - 3\} \\ 0 & \text{otherwise,} \end{cases}$$

$$k = 0, \dots, n-1.$$

Remaining questions

- ▶ An analog of the Hochschild–Serre spectral sequence.
- ▶ Commutative cohomology as a derived functor?
- ▶ Compute commutative cohomology for various “interesting” algebras.
- ▶ An analog of the Hopf formula for the second degree *homology*.
- ▶ Define the cup-product in the “standard” way.
- ▶ Algebras of cohomological dimension 1?
- ▶ Euler-Poincaré characteristic?
- ▶ Whether the variety of commutative Lie algebras is Schreier?
- ▶ ...

That's all. Thank you.