Hom-Lie and post-Lie structures on current and Kac-Moody algebras

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Based on:

- arXiv:1805.00187 (joint with Abdenacer Makhlouf)
- arXiv:1805.04267 (joint with Dietrich Burde)

These slides are at `http://www1.osu.cz/~zusmanovich/`

What is a Hom-Lie algebra?

A triple $(L, [\,\cdot\,,\cdot\,], \alpha)$, where $\alpha : L \to L$ is linear, and $[\,\cdot\,,\cdot\,]$ is anticommutative and satisfies the *Hom-Jacobi* identity:

$$[x, y] = -[y, x]$$

$$[[x, y], \alpha(z)] + [[z, x], \alpha(y)] + [[y, z], \alpha(x)] = 0$$

A little history

- Aizawa & Sato (1991): "Hom-Witt" $(W_1, [\cdot,\cdot], \alpha)$

$$[e_i, e_j] = \{i-j\}_q e_{i+j}, \text{ where } \{n\}_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$\alpha(e_i) = \frac{q^i + q^{-i}}{2} e_i$$

- Hartwig, Larsson & Silvestrov (2006): Hom-Lie algebras

…

- many people (2006-today): Hom-associative:

$$(xy)\alpha(z) = \alpha(x)(yz)$$

and Hom-everything

Hom-Lie structures

A problem: given a Lie algebra L, determine all α's turning it into a Hom-Lie algebra.

Done for:

- Finite-dimensional semisimples, $p = 0$ (Jin & Li 2008 and Xie, Jin & Liu 2015)
  Answer: nontrivial structures only in the case of $sl(2)$.
- Witt algebra, infinite-dimensional algebras of Cartan type, loop algebras (Xie & Liu 2017).
- Current algebras and affine Kac-Moody algebras (Makhlouf & Zusmanovich 2018)
  Answer for Kac-Moody: "almost trivial".

Hom-Lie structures in $p > 0$

Nothing is done in $p > 0$.

Conjecture

If a simple finite-dimensional Lie algebra, $p > 0$, admits a nontrivial Hom-Lie structure, then it is isomorphic either to a 3-dimensional simple algebra or to a (form of) Zassenhaus algebra.

Hom-Lie structures on tensor products

Theorem

Let $(\mathscr{P}, \mathscr{P}^!)$ be a (commutative, anticommutative) pair of binary quadratic Koszul dual operads, A an algebra over $\mathscr{P}$, B an algebra over $\mathscr{P}^!$. Then:

$$
\begin{aligned}
&\mathrm{HomLie}\left((A \otimes B)^{(-)}\right) \\
&\qquad \simeq \mathrm{HomLie}(A) \otimes \mathrm{HomCycl}(B) + \mathrm{Hom2Nilp}(A) \otimes \mathrm{End}(B) \\
&\qquad + \mathrm{HomCycl}(A) \otimes \mathrm{HomLie}(B) + \mathrm{End}(A) \otimes \mathrm{Hom2Nilp}(B).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{HomLie}(A) &= \text{the space of all Hom-Lie structures on } A: \\
&\quad (xy)\alpha(z) + (zx)\alpha(y) + (yz)\alpha(x) = 0 \\
\mathrm{HomCycl}(A) &= \text{the space of all Hom-cyclic structures on } A: \\
&\quad (xy)\alpha(z) = (zx)\alpha(y) \\
\mathrm{Hom2Nilp}(A) &= \text{the space of all Hom-2-nilpotent structures on } A: \\
&\quad (xy)\alpha(z) = 0
\end{aligned}
$$

When Hom-Lie structures form a Jordan algebra?

Observation 1 (Xie, Jin & Liu 2015, Xie & Liu 2017)

In all known cases so far, Hom-Lie structures on a Lie algebra form a Jordan algebra with respect to

$$\alpha * \beta = \frac{1}{2}(\alpha \circ \beta + \beta \circ \alpha)$$

Observation 2 (Makhlouf & Zusmanovich 2018)

For some current Lie algebras $L \otimes A$ this is not so.

Question

When Hom-Lie structures form a Jordan algebra? Which Jordan algebras arise in this way?

Commutative post-Lie structures

A problem: given a Lie algebra L, determine all bilinear products $\cdot : L \times L \to L$ such that:

$$\begin{aligned} x \cdot y &= y \cdot x \\ [x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z) \\ x \cdot [y, z] &= [x \cdot y, z] + [y, x \cdot z] \end{aligned}$$

Done for:

- finite-dimensional perfect algebras, $p = 0$ (Burde & Moens 2016)
  Answer: trivial
- free nilpotent algebras (Burde, Moens & Dekimpe 2017)
- Witt algebras and related (Tang 2017, Tang & Yang 2018)
  Answer: trivial
- loop algebras and affine Kac-Moody algebras (Burde & Zusmanovich 2018)
  Answer: "almost trivial"

That's all. Thank you.