Structure functions and Spencer cohomology in zero and positive characteristics

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Structure functions

Let $G \subseteq GL(n)$ be a real or complex Lie group, M an n-dimensional real or complex manifold.

A *G-structure* on M is a reduction of the principal $GL(n)$-bundle to the principal G-bundle.

Structure functions are obstructions to integrability (= local flattening) of M endowed with a G-structure.

Some (well known) particular cases:

G	**name of a G-structure**	**name of a structure function**
$O(n)$	Riemann metric	Riemann tensor
$O(n) \times \mathbb{R}^*$	almost conformal structure	Weyl tensor
$GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$	almost complex structure	Nijenhuis tensor

Spencer cohomology

Structure functions are interpreted in terms of the *Spencer cohomology* $H^*(\mathscr{L}_{-1}, \mathscr{L})$ of a graded Lie algebra $\mathscr{L} = \bigoplus_{n\geq -1} \mathscr{L}_n$.

Major examples of $\mathscr{L}$: Lie algebras of Cartan type W_n, S_n, H_{2n}.

Theorem (Serre)

The Spencer cohomology vanishes in degrees > 0 for W_n and S_n, and is fully computed for H_{2n}.

Spencer cohomology is also responsible for *filtered deformations* of a graded Lie algebra $\mathscr{L}$, and therefore important for characteristic $p > 0$ analogs of Lie algebras of Cartan type.

The algebras $\mathbb{W}(L, A)$

Let L be an abelian Lie algebra acting by derivations on an associative commutative algebra A, such that AL is a free submodule of $Der(A)$.

The Lie algebra $\mathbb{W}(L, A)$ is defined as the vector space $AL \simeq L \otimes A$ with multiplication

$$[x \otimes a, y \otimes b] = y \otimes ax(b) - x \otimes by(a).$$

The algebras $\mathbb{W}(L, A)$ (cont.)

Particular cases of the construction from the previous slide are:

1. $A = K[t_1, \dots, t_n]$, $L = \langle \frac{\mathrm{d}}{\mathrm{d}\,t_1}, \dots, \frac{\mathrm{d}}{\mathrm{d}\,t_n} \rangle$:
   $\mathbb{W}(L, A)$ = one-sided Jacobson–Witt algebra = infinite-dimensional Lie algebra of the general Cartan type W_n = Lie algebra of polynomial vector fields on the plane K^n.
2. $A = K[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, $L = \langle \frac{\mathrm{d}}{\mathrm{d}\,t_1}, \dots, \frac{\mathrm{d}}{\mathrm{d}\,t_n} \rangle$:
   $\mathbb{W}(L, A)$ = two-sided Jacobson–Witt algebra = Lie algebra of polynomial vector fields on the n-dimensional sphere.
3. K is of characteristic $p > 0$, $A = O(n; \overline{m})$, the algebra of divided powers in n variables with shearing parameters $\overline{m} = (m_1, \dots, m_n)$, $L = \langle \partial_1, \dots, \partial_n \rangle$:
   $\mathbb{W}(L, A)$ = finite-dimensional Lie algebra of the general Cartan type $W(n; \overline{m})$.

A unified approach to calculation of the Spencer cohomology: case W_n

Theorem

Let L has a basis $D_1, \ldots, D_n$ such that the algebra A decomposes as the tensor product of algebras $A_1 \otimes \cdots \otimes A_n$, with D_i acting on A_i. Then

$$\mathrm{H}^k(L, A) \simeq L \otimes \left(\bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} A_1^{D_1} \otimes \cdots \otimes (A_{i_1})_{D_{i_1}} \otimes \cdots \otimes (A_{i_k})_{D_{i_k}} \otimes \cdots \otimes A_n^{D_n} \right).$$

(at $i_1, \ldots, i_k$ are coinvariants, at the other places, invariants).

Sketch of the proof

1) $\mathrm{H}^k(L, \mathbb{W}(L, A)) \simeq L \otimes H^k(L, A)$.
2) Apply the Künneth formula.

A unified approach to calculation of the Spencer cohomology: case W_n (cont.)

Corollaries

Serre's vanishing result in $p = 0$, and non-vanishing result in $p > 0$. In particular,

$$\dim \mathrm{H}^k(W(n;\overline{m})_{-1}, W(n;\overline{m})) = n\binom{n}{k}.$$

Digression: Nijenhuis tensors

Theorem

The space of structure functions of a real $2n$-dimensional manifold endowed with a $GL(n,\mathbb{C})$-structure is $2n^2(n-1)$-dimensional.

Proof

For any associative commutative unital algebra A,

$$\mathrm{H}^2((W_n)_{-1}\otimes A, W_n\otimes A)\simeq \Big(B^{2,-1}((W_n)_{-1},W_n)\otimes \frac{S^2(A,A)}{A\oplus Der(A)}\Big) \oplus\Big(S^2((W_n)_{-1},(W_n)_{-1})\otimes \frac{C^2(A,A)}{\{\alpha\in C^2(A,A)\mid \alpha(a,b)=a\beta(b)-b\beta(a)\}}\Big).$$

Substitute $A=\mathbb{C}_{\mathbb{R}}$.

A unified approach to calculation of the Spencer cohomology: case S_n

The Lie algebra $\mathbb{S}(L, A)$ is defined as the kernel of homomorphism

$$div : \mathbb{W}(L, A) \to A, \quad x \otimes a \to x(a).$$

To compute the corresponding Spencer cohomology, apply the cohomology long exact sequence associated with the short exact sequence of L-modules

$$0 \to \mathbb{S}(L, A) \to \mathbb{W}(L, A) \stackrel{div}{\to} L(A) \to 0.$$

Corollary

$\mathrm{H}^k((S_n)_{-1}, S_n) = 0$ for $k > 0$.

The algebras $\mathbb{P}(A, \overline{D}, \overline{F})$

Let $\overline{D} = (D_1, \dots, D_n)$ and $\overline{F} = (F_1, \dots, F_n)$ be two n-element sets of pairwise commuting derivations of A. Then A equipped with the bracket

$$[a, b] = \sum_{i=1}^{n} \Big(D_i(a)F_i(b) - F_i(a)D_i(b) \Big),$$

denoted by $\mathbb{P}(A, \overline{D}, \overline{F})$, is a generalization of all kinds of Hamiltonian Lie algebras.

A unified approach to calculation of the Spencer cohomology: case H_{2n}

To compute the corresponding Spencer cohomology, apply considerations based on the Künneth formula, similar to the case of $\mathbb{W}(L, A)$ (but more cumbersome). Under similar assumptions,

$$\mathrm{H}^k\left(\mathbb{P}(A, \overline{D}, \overline{F})_{-1}, \mathbb{P}(A, \overline{D}, \overline{F})\right) \simeq \bigoplus_{k_1+\cdots+k_n=k} \mathrm{H}^{k_1}(A_1, \overline{D}_1, \overline{F}_1) \otimes \cdots \otimes \mathrm{H}^{k_n}(A_n, \overline{D}_1, \overline{F}_1).$$

For example, the number of different summands in the "classical" case, where each set of derivations $\overline{D}_i$, $\overline{F}_i$ consists of one element, and each $\mathbb{P}(A_i, \overline{D}_i, \overline{F}_i)_{-1}$ is 2-dimensional, the number of different summands in this formula is

$$\sum_{n_0+n_1+n_2=n,\ n_1+2n_2=k} \frac{n!}{n_0!n_1!n_2!},$$

where n_0, n_1, n_2 is the number of occurrences of 0th, 1st, and 2nd cohomology respectively.

That's all. Thank you.