Commutative cohomology in characteristic 2

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November 6, 2019

Commutative Lie algebras

Standing assumption: $p = 2$

$$[x, y] = [y, x]$$

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$$

$$\underset{[x,x]=0}{\text{Lie}} \quad \subset \quad \text{commutative Lie} \quad \subset \quad \text{Leibniz}$$

Commutative cohomology

$$0 \to \mathrm{S}^0(L, M) \xrightarrow{\mathrm{d}} \mathrm{S}^1(L, M) \xrightarrow{\mathrm{d}} \mathrm{S}^2(L, M) \xrightarrow{\mathrm{d}} \dots$$

$\mathrm{S}^n(L, M) =$ the space of n-linear *symmetric* maps $L \times \cdots \times L \to M$

$$\begin{aligned} \mathrm{d}\,\varphi(x_1, \dots, x_{n+1}) \\ &= \sum_{1 \le i < j \le n+1} \varphi([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{n+1}) \\ &+ \sum_{i=1}^{n+1} x_i \bullet \varphi(x_1, \dots, \widehat{x_i}, \dots, x_{n+1}). \end{aligned}$$

Why bother?

1) Operadic viewpoint: algebras should be defined by multilinear identities.
2) New phenomena in cohomology, similar to Lie superalgebras.
3) New invariant of (ordinary) Lie algebras.
4) Appears naturally in the context of classification of simple Lie algebras.

Some spectral sequences

(after Friedrich Wagemann)

- Hochschild–Serre spectral sequence

  $0 \to I \to L \to L/I \to 0$

  $E_2^{pq} = \mathrm{H}^p_{comm}(L/I, \mathrm{H}^q_{comm}(I, M)) \Rightarrow \mathrm{H}^{p+q}_{comm}(L, M)$

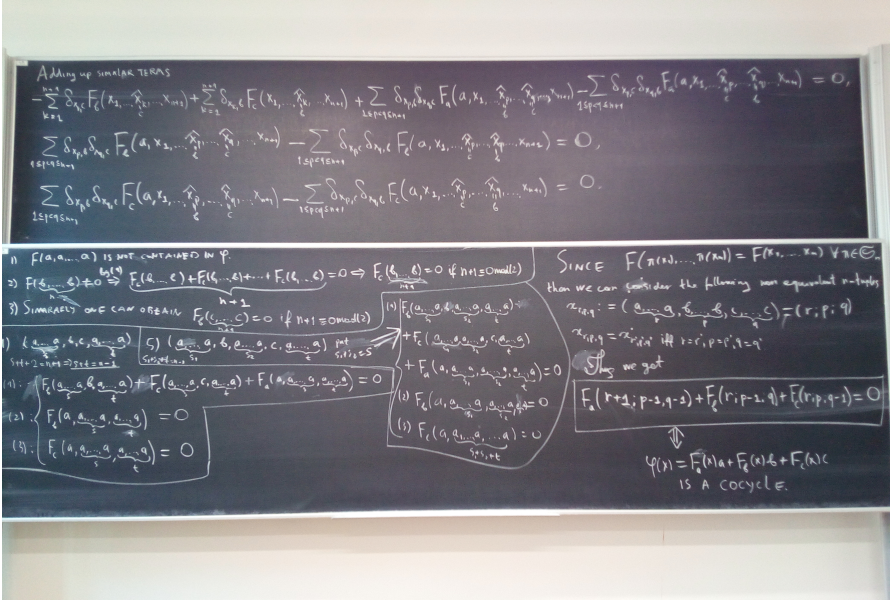
- Spectral sequences relating commutative cohomology with Chevalley–Eilenberg cohomology and with Leibniz cohomology

Cup product

$$(\varphi \smile \psi)(x_1, \ldots, x_{p+q}) = \sum_{\text{shuffles}} \varphi(x_{i_1}, \ldots, x_{i_p}) \cdot \psi(x_{j_1}, \ldots, x_{j_q})$$

$\smile$ turns $\mathrm{H}^*_{comm}(L, K)$ into an associative graded ring.

Some computations



2-dimensional nonabelian

$$L = \langle a, b \mid [a, b] = a \rangle$$

$$\chi_{pq}(\underbrace{a, \ldots, a}_{r}, \underbrace{b, \ldots, b}_{s}) = \begin{cases} 1 & \text{if } p = r \text{ and } q = s \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{H}^n_{comm}(L, K) \simeq \langle \chi_{pq} \mid p + q = n, p \text{ even} \rangle$$

$$\chi_{pq} \smile \chi_{rs} = \binom{p+r}{p} \binom{q+s}{q} \chi_{p+r,q+s}$$

Zassenhaus

$$W_1'(n) = \langle e_i = x^{(i+1)}\partial \mid -1 \le i \le 2^n - 3 \rangle$$

$$\dim \mathrm{H}^2_{comm}(W_1'(n), K) = n$$

Basic cocycles:

$$e_i \vee e_j \mapsto \begin{cases} 1 & \text{if } i = j = 2^k - 2, \text{or } \{i, j\} = \{-1, 2^{k+1} - 3\} \\ 0 & \text{otherwise} \end{cases},$$

$k = 0, \ldots, n - 1.$

Remaining questions

- Commutative cohomology as a derived functor?
- Compute commutative cohomology for various “interesting” algebras.
- An analog of the Hopf formula for the second degree *homology*.
- Define the cup-product in the “standard” way.
- Algebras of cohomological dimension 1?
- Euler-Poincaré characteristic?
- Whether the variety of commutative Lie algebras is Schreier?
- ...

References:

- V. Lopatkin and P. Zusmanovich, arXiv:1907.03690
- F. Wagemann, arXiv:1908.06764

That's all. Thank you.