Cohomology of current algebras

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L - Lie algebra
A - associative commutative algebra

Current Lie algebra is a vector space $L \otimes A$ under the bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

where $x, y \in L$, $a, b \in A$.

What current Lie algebras are good for?

Kac-Moody algebras

are central extensions of current Lie algebras $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$:

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}t\frac{d}{dt} + \mathbb{C}z$$

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y)Res\frac{df}{dt}g\ z$$

where $x, y \in \mathfrak{g},\ f, g \in \mathbb{C}[t, t^{-1}]$

$(\,\cdot\,, \cdot\,)$ is the Killing form on $\mathfrak{g}$.

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Modular semisimple Lie algebras

$$S \otimes K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p) + 1 \otimes D$$

Question

- How $H^*(L \otimes A, K)$ is expressed through invariants of L and A?
- How $H^*(L \otimes A, M \otimes V)$ is expressed through invariants of (L, M) and (A, V)?

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Applications

- degree 2 (deformations and central extensions): structure theory of modular Lie algebras, physics.
- degree 3: 2-Lie algebras and "physics" again.
- all degrees: combinatorial identities.

An elementary observation

Cocycles of the form

$$\Phi(x_1 \otimes a_1, \ldots, x_n \otimes a_n) = \varphi(x_1, \ldots, x_n) \otimes a_1 \cdots a_n \bullet v$$

for some $v \in V$ give rise to

$$H^*(L, M) \otimes V \subseteq H^*(L \otimes A, M \otimes V).$$

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Another elementary observation

Cocycles of the form

$$\Phi(x_1 \otimes a_1, x_2 \otimes a_2) = [x_1, x_2] \otimes F(a_1, a_2)$$

give rise to

$$Har^2(A, V) \subseteq H^2(L \otimes A, L \otimes V).$$

A naive desire

$$H^*(L \otimes A, M \otimes V) \simeq \bigoplus_i \mathcal{F}_i(L, M) \otimes \mathcal{G}_i(A, V)$$

for some functors $\mathcal{F}_i$ and $\mathcal{G}_i$.

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It fails miserably in general.

What is known about (co)homology of current Lie algebras?

- L is classical simple, A close to polynomial: Feigin, Garland & Lepowsky, Hanlon.
- L is algebra of infinite matrices, A arbitrary: additive K-theory Loday & Quillen, Feigin & Tsygan.
- L, A (almost) arbitrary, (co)homology of low degree.

Feigin, 1970-1990s:

An example

$H_*(\mathfrak{g} \otimes \mathbb{C}[t], \mathbb{C}) \simeq H_*(\mathfrak{g}, \mathbb{C})$
Tool: comparison of spectral sequences arising from the triangular decomposition $\mathfrak{g} \otimes \mathbb{C}[t] = \widehat{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \widehat{\mathfrak{n}}_+$ and using Kostant-like results about $H_*(\widehat{\mathfrak{n}}_+, \mathbb{C})$.

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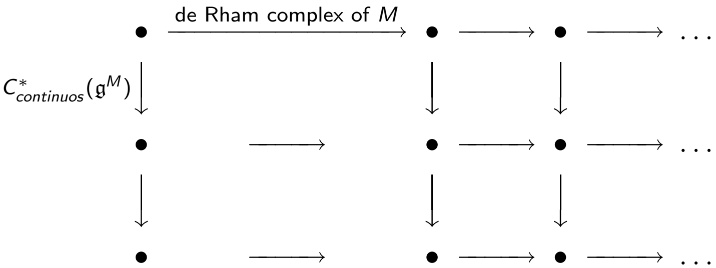
Another example

Partial results about $H^*_{continuos}(\mathfrak{g}^M)$.

Tool: a map from the Weil complex $\bigwedge^*(\mathfrak{g}) \otimes S^*(\mathfrak{g})$ to a certain bicomplex



Garland & Lepowsky, 1976: $H^*(\mathfrak{g} \otimes t\mathbb{C}[t], \mathbb{C})$
Hanlon, 1986: $H^*(\mathfrak{g} \otimes \mathbb{C}[t]/(t^n), \mathbb{C})$
Tool: eigenvectors of the Laplacian (Gelfand–Fuchs style).

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Tsygan, 1983 and Loday & Quillen, 1984:

$$H_*(gl(A), K) \simeq \bigwedge(HC_*(A))$$

Lesson: cyclic (co)homology is involved!

How cyclic cohomology appears in current Lie algebras cohomology?

First degree cyclic cohomology:
$HC^1(A) = \{\alpha : A \times A \to K \mid \alpha(ab,c) + \alpha(ca,b) + \alpha(bc,a) = 0\}$

Let $\varphi \otimes \alpha \in Z^2(L \otimes A, K)$, $\varphi : L \times L \to K$, $\alpha : A \times A \to K$:

$$\begin{aligned} &\varphi([x,y],z) \otimes \alpha(ab,c) \\ +\, &\varphi([z,x],y) \otimes \alpha(ca,b) \\ +\, &\varphi([y,z],x) \otimes \alpha(bc,a) = 0 \end{aligned}$$

for any $x,y,z \in L$, $a,b,c \in A$.
Cyclically permute x,y,z and sum up the 3 equalities obtained:

$$\Big(\varphi([x,y],z) + \varphi([z,x],y) + \varphi([y,z],x)\Big) \otimes \Big(\alpha(ab,c) + \alpha(bc,a) + \alpha(ca,b)\Big) = 0.$$

Low degree cohomology

Nice formulae:

- $H^1(L \otimes A, M \otimes V) \simeq H^1(L, M) \otimes V + Hom_L(L, M) \otimes Der(A, V)$
  (Zusmanovich, 2005)
- $H^2(L \otimes A, K) \simeq H^2(L, K) \otimes A^* + B(L) \otimes HC^1(A)$
  (Haddi, 1992 and Zusmanovich, 1994)
  (both assuming $[L, L] = L$)
- $H^2(\mathfrak{g} \otimes A, \mathfrak{g} \otimes A) \simeq Har^2(A, A)$ (Cathelineau, 1987)
- If $W_1(n)$ is the modular Zassenhaus algebra, then

  $$\begin{aligned} & H^2(W_1(n) \otimes A, W_1(n) \otimes A) \\ \simeq\ & H^2(W_1(n), W_1(n)) \otimes A + Der(A) + Der(A) + Har^2(A, A) \end{aligned}$$

  (Zusmanovich, 2003)
- $H^3(\mathfrak{g} \otimes A, K) \simeq HC^2(A)$ or $HD^2(A)$ (Cathelineau, 1987)

Low degree cohomology

... and ugly formulae:
some part (not all!) of $H^2(L \otimes A, M \otimes V)$ is isomorphic to

$$H^2(L,M) \otimes V + H^2_M(L) \otimes \frac{Hom(A,V)}{V \oplus Der(A,V)} + \mathscr{H}(L,M) \otimes Der(A,V) + \mathscr{B}(L,M) \otimes \frac{Har^2(A,V)}{\mathscr{P}_+(A,V)}$$
$$+ C^2(L,M)^L \otimes \mathscr{P}_+(A,V) + \mathscr{X}(L,M) \otimes \frac{\mathscr{A}(A,V)}{\mathscr{P}_+(A,V)} + \mathscr{T}(L,M) \otimes \frac{D(A,V)}{Der(A,V)}$$
$$+ Poor_-(L,M) \otimes \frac{S^2(A,V)}{Hom(A,V) + D(A,V) + Har^2(A,V) + \mathscr{A}(A,V)}$$

where:
$d^{[\,]}\varphi(x,y,z) = \varphi([x,y],z) + \curvearrowright$;
$\wp\alpha(a,b,c) = \alpha(ab,c) + \curvearrowright$;
$D\alpha(a,b,c) = a \bullet \alpha(b,c) + \curvearrowright$;
$\mathscr{B}(L,M) = \{\varphi \in C^2(L,M) \mid \varphi([x,y],z) + z \bullet \varphi(x,y) = 0; d^{[\,]}\varphi(x,y,z) = 0\}$;
$Q^2(L,M) = \{d\psi \mid \psi \in Hom(L,M); x \bullet \psi(y) = y \bullet \psi(x)\}$;
$H^2_M(L) = (Z^2(L,M^L) + Q^2(L,M))/Q^2(L,M)$;
$\mathscr{J}(L,M) = \{\varphi \in C^2(L,M) \mid \varphi(x,y) = \psi([x,y]) - \frac{1}{2}x \bullet \psi(y) + \frac{1}{2}y \bullet \psi(x)$ for $\psi \in Hom(L,M)\}$;
$\mathscr{H}(L,M) = (\mathscr{K}(L,M) + \mathscr{J}(L,M))/\mathscr{J}(L,M)$.
$\mathscr{X}(L,M) = \{\varphi \in C^2(L,M) \mid 2\varphi([x,y],z) = z \bullet \varphi(x,y); \varphi([x,y],z) = \varphi([z,x],y)\}$;
$\mathscr{T}(L,M) = \{\varphi \in C^2(L,M) \mid 3\varphi([x,y],z) = 2z \bullet \varphi(x,y); \varphi([x,y],z) = \varphi([z,x],y)\}$;
$Poor_-(L,M) = \{\varphi \in C^2(L,M^L) \mid \varphi([L,L],L) = 0\}$;
$D(A,V) = \{\beta \in Hom(A,V) \mid \beta(abc) = a \bullet \beta(bc) - bc \bullet \beta(a) + \curvearrowright\}$;
$\mathscr{P}_+(A,V) = \{\alpha \in S^2(A,V) \mid \alpha(ab,c) = a \bullet \alpha(b,c) + b \bullet \alpha(a,c)\}$;

$\mathscr{A}(A,V) = \{\alpha \in S^2(A,V) \mid 2D\alpha = \wp\alpha\}$.

How to (methodically) “compute” cohomology of general current Lie algebras?

Cauchy formula:

$$\bigwedge^n (L \otimes A) \simeq \bigoplus_{\lambda \vdash n} Y_\lambda(L) \otimes Y_{\lambda^\sim}(A)$$

Y_λ - *Schur functor* associated with the Young diagram λ.

Examples:

$Y_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \frac{1}{3!}\sum_{\sigma\in S_3}(-1)^\sigma \sigma$

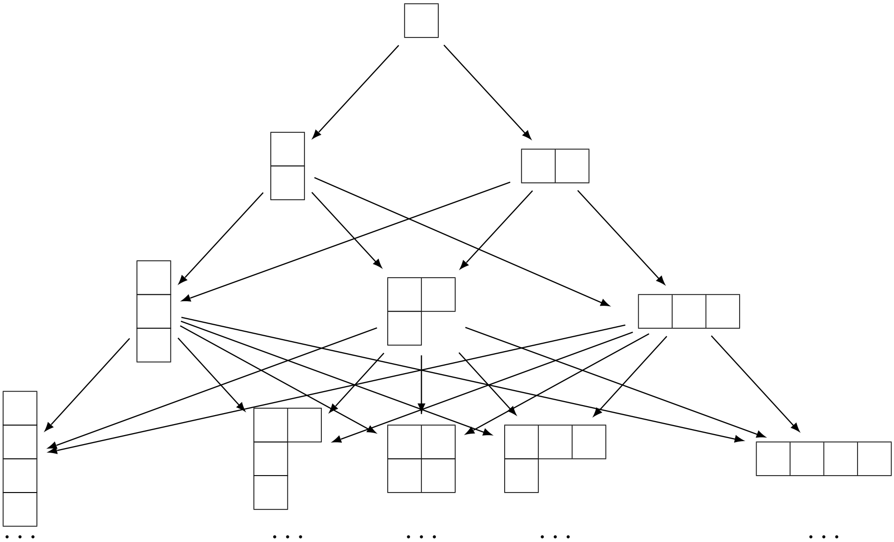
$Y_{\square\square\square} = \frac{1}{3!}\sum_{\sigma\in S_3} \sigma$

$Y_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = \frac{1}{3}(e + (12) - (13) - (123))$

$\lambda^\sim$ - obtained from λ by interchanging rows and columns

How Young symmetrizers interact with the differential?

each Young diagram λ represents $Hom(Y_\lambda(L), M) \otimes Hom(Y_\lambda(A), V)$



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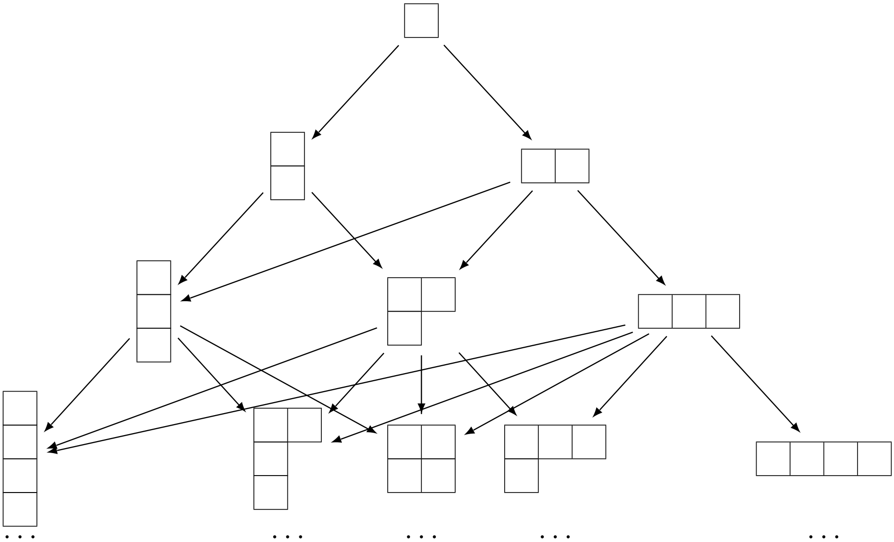
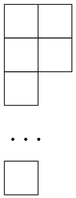


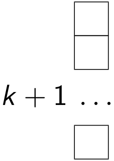
Which arrows do not vanish?

- going from “right” to “left”
- the source Young diagram included into the target Young diagram
- the target Young diagram is of the shape:



A filtration and a spectral sequence

$F^k C^*$ = "closure" under non-vanishing arrows of



That's all. Thank you.

Slides at `http://justpasha.org/math/spb-2011-algsem.pdf`