Non-Koszulity of the alternative operad and inversion of polynomials

Pasha Zusmanovich

April 19, 2011

based on joint work with Askar Dzhumadil'daev
arXiv:0906.1272

What is an operad?

An *operad* is a sequence $\mathcal{P}(n)$ of right S_n-modules equipped with compositions

$$\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n+m-1)$$

satisfying associativity-like conditions:

$$(f \circ_i g) \circ_j h = f \circ_j (g \circ_{i-j+1} h)$$

- J. Stasheff, *What is... an operad?*, Notices Amer. Math. Soc. June/July 2004.
- P. Cartier, *What is an operad?*, The Independent Univ. of Moscow Seminars, 2005.

What is an operad?

An *operad* is a sequence $\mathcal{P}(n)$ of right S_n-modules equipped with compositions

$$\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n+m-1)$$

satisfying associativity-like conditions:

$$(f \circ_i g) \circ_j h = f \circ_j (g \circ_{i-j+1} h)$$

- J. Stasheff, *What is... an operad?*, Notices Amer. Math. Soc. June/July 2004.
- P. Cartier, *What is an operad?*, The Independent Univ. of Moscow Seminars, 2005.

Primitive view: multilinear parts of relatively free algebras.
$\mathcal{P}(n)$ = space of multilinear (nonassociative) polynomials of degree n.

What is a Koszul operad?

Koszulity = "good" homological properties.
Associative, Lie and associative commutative operads are Koszul.

What is a Koszul operad?

Koszulity = "good" homological properties.
Associative, Lie and associative commutative operads are Koszul.

Ginzburg–Kapranov criterion

If a binary quadratic operad $\mathcal{P}$ over a field of characteristic zero is Koszul, then

$$g_{\mathcal{P}}(g_{\mathcal{P}^!}(t)) = t$$

where

$$g_{\mathcal{P}}(t) = \sum_{n=1}^{\infty} (-1)^n \frac{\dim \mathcal{P}(n)}{n!} t^n$$

is the Poincaré series of $\mathcal{P}$, and $\mathcal{P}^!$ is the operad dual to $\mathcal{P}$.

Examples of Poincaré series

$$g_{\mathcal{A}ss}(t) = \sum_{n=1}^{\infty} (-1)^n \frac{n!}{n!} t^n = -\frac{t}{1+t}$$

$$g_{\mathcal{C}omm}(t) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} t^n = e^{-t} - 1$$

$$g_{\mathcal{L}ie}(t) = \sum_{n=1}^{\infty} (-1)^n \frac{(n-1)!}{n!} t^n = -log(1+t)$$

What is a dual operad?

Pairing $\langle\, ,\, \rangle$ on the space of multilinear (nonassociative) polynomials of degree 3:

$$\left\langle (x_i x_j)x_k, (x_{\sigma(i)}x_{\sigma(j)})x_{\sigma(k)} \right\rangle = (-1)^\sigma$$

$$\left\langle x_i(x_j x_k), x_{\sigma(i)}(x_{\sigma(j)}x_{\sigma(k)}) \right\rangle = -(-1)^\sigma$$

$$\left\langle (x_i x_j)x_k, x_{i'}(x_{j'}x_{k'}) \right\rangle = 0$$

$\sigma \in S_3$.

R - relations in a binary quadratic operad
$R^!$ - dual space of relations under this pairing

Dual operads

Examples

$\mathcal{A}ss^! = \mathcal{A}ss$

$\mathcal{L}ie^! = \mathcal{C}omm$

Dual operads

Examples

$\mathcal{A}ss^! = \mathcal{A}ss$
$\mathcal{L}ie^! = \mathcal{C}omm$

A remarkable fact

If A, B are algebras over operads dual to each other, then $A \otimes B$ under the bracket

$$[a \otimes b, a' \otimes b'] = aa' \otimes bb' - a'a \otimes b'b$$

for $a, a' \in A$, $b, b' \in B$, is a Lie algebra.

Alternative algebras

$$(xy)y = x(yy)$$

$$(xx)y = x(xy)$$

An example: the 8-dimensional octonion algebra.

Alternative algebras

$$(xy)y = x(yy)$$
$$(xx)y = x(xy)$$

An example: the 8-dimensional octonion algebra.

Theorem

The alternative operad over a field of characteristic zero is not Koszul.

Alternative algebras

$$(xy)y = x(yy)$$
$$(xx)y = x(xy)$$

An example: the 8-dimensional octonion algebra.

Theorem

The alternative operad over a field of characteristic zero is not Koszul.

Proof by the Ginzburg–Kapranov criterion.

Proof of the Theorem

Easy part:
Dual alternative algebra: associative and $x^3 = 0$.

$$g_{\mathcal{A}lt^!}(t) = -t + t^2 - \frac{5}{6}t^3 + \frac{1}{2}t^4 - \frac{1}{8}t^5.$$

Proof of the Theorem

Easy part:
Dual alternative algebra: associative and $x^3 = 0$.

$$g_{\mathcal{A}lt^!}(t) = -t + t^2 - \frac{5}{6}t^3 + \frac{1}{2}t^4 - \frac{1}{8}t^5.$$

Difficult part:

$$g_{\mathcal{A}lt}(t) = -t + t^2 - \frac{7}{6}t^3 + \frac{4}{3}t^4 - \frac{35}{24}t^5 + \frac{3}{2}t^6 + O(t^7).$$

Proof of the Theorem

Easy part:
Dual alternative algebra: associative and $x^3 = 0$.

$$g_{\mathcal{A}lt^!}(t) = -t + t^2 - \frac{5}{6}t^3 + \frac{1}{2}t^4 - \frac{1}{8}t^5.$$

Difficult part:

$$g_{\mathcal{A}lt}(t) = -t + t^2 - \frac{7}{6}t^3 + \frac{4}{3}t^4 - \frac{35}{24}t^5 + \frac{3}{2}t^6 + O(t^7).$$

Not Koszul by Ginzburg–Kapranov:

$$g_{\mathcal{A}lt}(g_{\mathcal{A}lt^!}(t)) = t - \frac{11}{72}t^6 + O(t^7).$$

Albert

$\dim \mathcal{A}lt(n)$ for $n = 1, \ldots, 6$ are computed with the help of *Albert* (developed in 1990s by David Pokrass Jacobs) and PARI/GP.

`http://justpasha.org/math/albert/`

Questions

Question

Does the inverse of the polynomial

$$g_{\mathcal{A}lt^!}(t) = -t + t^2 - \frac{5}{6}t^3 + \frac{1}{2}t^4 - \frac{1}{8}t^5$$

have alternating signs?

Questions

Question

Does the inverse of the polynomial

$$g_{\mathcal{A}lt^!}(t) = -t + t^2 - \frac{5}{6}t^3 + \frac{1}{2}t^4 - \frac{1}{8}t^5$$

have alternating signs?

Another question (Martin Markl and Elizabeth Remm, 2009–2011)

Does the inverse of the polynomial

$$-t + t^8 - t^{15}$$

have alternating signs?

Three morals of this story

- One can do something in operads without really understanding them.
- Use open source. Make your software publically available.
- Questions about signs of inversions of polynomials are difficult. Study them!

Three morals of this story

- One can do something in operads without really understanding them.
- Use open source. Make your software publically available.
- Questions about signs of inversions of polynomials are difficult. Study them!

That's all. Thank you.

Slides at `http://justpasha.org/math/alternative/`