NON-KOSZULITY OF THE ALTERNATIVE OPERAD AND INVERSION OF POLYNOMIALS (EXTENDED ABSTRACT)

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Operads are in fashion nowadays. *Koszulity* is an important property of operads, saying, roughly, that given operad possesses nice homological properties. Operads governing the behavior of all major classes of algebras – associative, Lie, Poisson, etc. – are Koszul, what explains, retroactively, why these classes of algebras have nice and reach (co)homological theories.

It is naturally a popular occupation these days to establish whether a given operad is Koszul or not. In doing so, one may use a *Ginzburg–Kapranov criterion* that says that if an operad $\mathcal{P}$ defined over a field of characteristic zero and satisfying some natural restrictions, is Koszul, then

$$g_{\mathcal{P}}(g_{\mathcal{P}^!}(t)) = t, \tag{1}$$

where

$$g_{\mathcal{P}}(t) = \sum_{n=1}^{\infty} (-1)^n \frac{\dim \mathcal{P}(n)}{n!} t^n$$

is the Poincaré series of the operad $\mathcal{P}$, and $\mathcal{P}^!$ is the Koszul dual of $\mathcal{P}$.

So, if we are (un)lucky, and the equality (1) does not hold for an operad $\mathcal{P}$, then $\mathcal{P}$ is not Koszul. We are using this approach to establish a non-Koszulity of the operad $\mathcal{A}lt$ governing the behavior of alternative algebras, answering a question of Loday. *Alternative algebras* are generalization of associative algebras and are determined by two identities of "weak associativity":

$$(xy)y = x(yy)$$
$$(xx)y = x(xy).$$

An example of a non-associative alternative algebra is the famous octonion algebra.

Some authors expressed a viewpoint that non-Koszulity is a rather pathological property, and all "occuring in the real life" algebras should be algebras over a Koszul operad. As we see, alternative algebras provide a "real life" example violating this principle.

So, first, we need to compute identities defining an operad dual to the alternative one. This is a matter of a simple exercise, and the corresponding identities are, first, associativity, and, second, $x^3 = 0$.

Next we should compute enough first terms in the dimension sequence $\dim \mathcal{P}(n)$ for the alternative and dual alternative operads, to achieve the violation of (1). These terms are nothing but dimensions of the multilinear components of the free algebras of rank n in the corresponding varieties of algebras. So, from now on, we can forget about operads and think in much more pedestrian terms.

For the dual alternative algebras, i.e. for associative algebras satisfying the identity $x^3 = 0$, this is easy: the corresponding Poincaré series is actually a polynomial

$$g_{\mathcal{A}lt^!}(t) = -t + t^2 - \frac{5}{6}t^3 + \frac{1}{2}t^4 - \frac{1}{8}t^5. \tag{2}$$

Free alternative algebras are much more difficult objects than, for example, their associative or Lie counterparts, and are still not understood sufficiently well, and to compute

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the corresponding dimension sequence is much more difficult. For this, we utilize a program *Albert* written at the beginning of 1990s by the team from Clemson University lead by David Pokrass Jacobs. This is a remarkably useful program for dealing with identities of algebras, and, in particular, for computations of the corresponding dimensions. A modified version of this program, suited for our needs, is located at `http://justpasha.org/math/albert/` .

Computations of such things over the field of rational numbers, due to explosive growth of numerators and denominators occuring at intermediate stages of computations, seem to be unfeasible, and Albert computes over prime fields. Juggling with Chinese-remainder-type arguments, we produce the number of primes sufficient to deduce the result in characteristic zero, and perform computations for all these primes. The beginning terms of the corresponding Poincaré series are

$$g_{\mathcal{A}lt}(t) = -t + t^2 - \frac{7}{6}t^3 + \frac{4}{3}t^4 - \frac{35}{24}t^5 + \frac{3}{2}t^6 + O(t^7),$$

and

$$g_{\mathcal{A}lt}(g_{\mathcal{A}lt^!}(t)) = t - \frac{11}{72}t^6 + O(t^7),$$

what violates (1).

One may try to avoid difficult computations of $g_{\mathcal{A}lt}(t)$, and argue as follows. Look at the inverse of polynomial (2). If, at certain place, the consecutive terms of this inverse will have the same sign, then this inverse cannot be a Poincaré series of any operad, and we get contradiction with (1). Interestingly enough, this does not seem to be the case for up to the first 1000 terms: the signs are really alternating.

Question. *Does the inverse of the polynomial (2) have alternating signs? If yes, what combinatorial interpretation this may have?*

The same kind of questions arise in similar contexts, for example in [MR] the same is asked about the polynomial $-t + t^8 - t^{15}$. Such questions seem to be, somewhat surprisingly, difficult.

Based on the work with Askar Dzhumadil'daev [DZ].

References

[DZ] A. Dzhumadil'daev and P. Zusmanovich, *The alternative operad is not Koszul*, Experiment. Math., to appear; arXiv:0906.1272 (a more recent version may be available at `http://justpasha.org/math/`).

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