On simple 15-dimensional Lie algebras in characteristic 2

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https://web.osu.cz/∼Zusmanovich/papers/15dim.pdf

Classification of simple Lie algebras

$p = 0$:
Killing, Cartan, Dynkin, ...

$p > 3$:
Witt, Kostrikin, Shafarevich, Block, Wilson, Premet, Strade, ...

$p = 2, 3$:
?

Tori, toral rank

Toral elements in the p-envelope: $x^{[p]} = x$.

Torus: A subalgebra consisting of toral elements.

(Absolute) toral rank: Maximal dimension of a torus.

Example ($p = 2$)

The simple 3-dimensional Lie algebra S

$$[e, h] = e, \quad [f, h] = f, \quad [e, f] = h$$

has absolute toral rank 2: $\langle h + e + e^{[2]}, h + f + f^{[2]} \rangle$ is a torus.

The 15-dimensional Skryabin algebra

From now on, we assume $p = 2$.

A simple 15-dimensional algebra defined by Skryabin (1998) appeared in the classification of simple Lie algebras of absolute toral rank 2 having a Cartan subalgebra of toral rank 1 (2015, Grishkov & Zusmanovich).

It is a deformation of a semisimple Lie algebra

$$\mathsf{S} \otimes \mathcal{O}_1(2) + f^{[2]} \otimes \langle 1, x \rangle + \partial,$$

where $\mathcal{O}_1(2)$ is the 4-dimensional divided powers algebra over an indeterminate x, ∂ its standard derivation.

The 15-dim. Skryabin algebra, multiplication table

	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	c_1	c_2	c_3	c_4	c_5	d
b_1	b_3	b_1	βc_5	b_6	b_4	δc_4	b_9	b_7	$\delta c_5 + d$	c_3	c_1	b_2	b_5	βc_2
b_2		b_2	b_6	0	b_5	b_9	0	b_8	c_3	0	c_2	0	0	0
b_3			b_4	b_5	0	b_7	b_8	0	c_1	c_2	0	0	0	0
b_4				0	0	$\delta c_5 + d$	c_3	c_1	b_3	c_4	b_2	b_5	0	b_1
b_5					0	c_3	0	c_2	c_4	0	0	0	0	b_2
b_6						c_1	c_2	0	0	0	c_4	0	0	b_3
b_7							0	0	b_6	c_5	b_5	b_8	c_2	b_4
b_8								0	c_5	0	0	0	0	b_5
b_9									0	0	c_5	0	0	b_6
c_1										0	b_8	c_2	0	b_7
c_2											0	0	0	b_8
c_3												0	0	b_9
c_4													0	0
c_5														c_4

The 15-dim. Skryabin algebra, some properties

- $\mathrm{H}^2(L, K) = 0$
- $\dim \mathrm{H}^3(L, K) = 15$
- $\dim \mathrm{H}^2(L, L) = 13$
- Derivation algebra = 2-envelope is of dimension 19.
- No nontrivial symmetric bilinear forms.
- The sandwich subalgebra is of dimension 3.
- The absolute toral rank is 4.
- Lots of 7-dimensional simple subalgebras, each isomorphic either to the Zassenhaus algebra, or to a Hamiltonian algebra.
- $Aut(L)$ is isomorphic to the semidirect product of $K^\times$ and a 7-dimensional unipotent algebraic group.
- Lots of gradings (over $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^4$, ...).

Thin decomposition

Let $\dim L = 2^n - 1$, the absolute toral rank is n, T is a torus of dimension n in the 2-envelope such that $L \cap T = 0$. Let the root space decomposition of L with respect to T is of the form

$$L = \bigoplus_{\substack{\alpha \in GF(2)^n \\ \alpha \neq (0,\dots,0)}} \langle e_\alpha \rangle.$$

Such decomposition is called *thin*.
The multiplication table in the basis $\{e_\alpha\}$ is of the form

$$[e_\alpha, e_\beta] = \begin{cases} e_{\alpha+\beta} \\ 0 \end{cases}$$

Fact

Over GF(2), the Skryabin algebra admits a thin decomposition with respect to each of the $26,880$ 4-dimensional tori in the 2-envelope.

Comparison with Eick's list

Eick (2010) produced a computer-generated list of simple Lie algebras over GF(2) of dimension ≤ 20. There are 8 15-dimensional algebras in the list, and the Skryabin algebra is not there!

Some further questions

- Classify simple Lie algebras admitting a thin decomposition.
- Prove that in the “most” of the cases a simple Lie algebra contains a simple 7-dimensional subalgebra.

That's all. Thank you.