Octonion matrix algebras

Pasha Zusmanovich

University of Ostrava

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Octonion matrix algebras

$S^+(M_n(\mathbb{O}), J) = \{A \in M_n(\mathbb{O}) \mid J(A) = A\}$ Hermitian

$S^-(M_n(\mathbb{O}), J) = \{A \in M_n(\mathbb{O}) \mid J(A) = -A\}$ skew-Hermitian

$J : (a_{ij}) \mapsto (\overline{a_{ji}})$ an involution

Why bother?

$\boldsymbol{S^+(M_n(\mathbb{O}), J)}$:

$n = 1$: the ground field K

$n = 2$: the 10-dimensional simple Jordan algebra of symmetric bilinear form

$n = 3$: the famous 27-dimensional exceptional simple Jordan algebra

$n \geq 4$: no longer Jordan, but appear in M-theory

$\boldsymbol{S^-(M_n(\mathbb{O}), J)}$:

$n = 1$: the 7-dimensional simple Malcev algebra $\mathbb{O}^-$.

Some references

- M. Bremner and I. Hentzel, *Identities for algebras of matrices over the octonions*, J. Algebra **277** (2004), no.1, 73–95.
- P. Jordan, *Zur Theorie nicht-assoziativer Algebren*, Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl. 1968, no.2, 27–38.
- J. Lukierski and F. Toppan, *Generalized space-time supersymmetries, division algebras and octonionic M-theory*, Phys. Lett. B **539** (2002), no.3-4, 266–276.
- H. Petyt, *Derivations of octonion matrix algebras*, Comm. Algebra **47** (2019), no.10, 4216–4223.
- H. Rühaak, *Matrix-Algebren über einer nicht-ausgearteten Cayley-Algebra*, PhD Thesis, Univ. Hamburg, 1968.

Simplicity

Theorem 1

The algebras $S^{+}(M_n(\mathbb{O}), J)$ and $S^{-}(M_n(\mathbb{O}), J)$ are simple.

Method of the proof: use realizations

$$S^{\pm}(M_n(\mathbb{O}), J) \simeq M_n^{\pm}(K) \otimes 1 + M_n^{\mp}(K) \otimes \mathbb{O}^{-},$$

and a variant of the Jacobson density theorem.

δ-derivations and associative forms

$$D(xy) = \delta D(x)y + \delta x D(y)$$

Theorem 2

δ-derivations of $S^{+}(M_n(\mathbb{O}), J)$ and $S^{-}(M_n(\mathbb{O}), J)$ are trivial (i.e., either the usual derivations, or multiplications by a scalar).

(Earlier derivations were computed by Petyt).

Theorem 3

Symmetric associative forms on $S^{+}(M_n(\mathbb{O}), J)$ and $S^{-}(M_n(\mathbb{O}), J)$ are:

$$(X, Y) \mapsto \mathrm{Tr}\,(XY + \overline{X}\,\overline{Y}).$$

Further questions

- Automorphisms? Conjecture: $G_2 \times SO(n)$.
- Identities?
- Subalgebras?
- Deformations?

That’s all. Thank you.