When Hom-Lie structures form a Jordan algebra

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https://web.osu.cz/∼Zusmanovich/papers/homlie-jordan.pdf

Hom-Lie structures

A Hom-Lie structure on a Lie algebra L is a linear map $\varphi : L \to L$ such that

$$[[x, y], \varphi(z)] + [[z, x], \varphi(y)] + [[y, z], \varphi(x)] = 0.$$

$\mathrm{HomLie}(L)$ – the vector space of all Hom-Lie structures on L.

Hom-Lie structures on "interesting" Lie algebras

- Classical simple: trivial except of $sl(2)$ (Jin–Li 2008, Xie–Jin–Liu 2015)
- Kac–Moody: "almost" trivial (Makhlouf–Zusmanovich 2018)
- Lie algebras of vector fields: trivial except of the Witt algebra (Xie–Liu 2017)
- Generalized Witt algebra W_G, $G \subseteq F^\times$,

$$[e_\alpha, e_\beta] = (\beta - \alpha) e_{\alpha+\beta}$$

HomLie$(W_G) \simeq K[G]$, spanned by $e_\alpha \mapsto e_{\alpha+\sigma}$.

Hom-Lie structures as a Jordan algebra

Observation

In all these cases, HomLie(L) is closed with respect to the "Jordan product"

$$\frac{1}{2}(\varphi \circ \psi + \psi \circ \varphi)$$

Question

For which Lie algebras L this is true?

Answer

Not for all L.

An example

$$L = \langle x, y \mid [x, y] = x \rangle \otimes K[t]$$

$$\varphi \otimes 1, \psi \otimes \alpha \in \mathrm{HomLie}(L)$$

$$(\varphi \otimes 1) \circ (\psi \otimes \alpha) + (\psi \otimes \alpha) \circ (\varphi \otimes 1) \notin \mathrm{HomLie}(L)$$

$$\varphi : \begin{array}{l} x \mapsto y \\ y \mapsto 0 \end{array} \qquad \psi : \begin{array}{l} x \mapsto x \\ y \mapsto 0 \end{array}$$

$$\alpha(t^i) = t^{2i}$$

A criterion

The following are equivalent:

- HomLie(L) is a Jordan algebra;
- HomLie(L) is closed with respect to squares;
- HomLie(L) is closed with respect to polynomials;
- For any $\varphi, \psi \in$ HomLie(L),

$$[F(x,y),z] + [F(z,x),y] + [F(y,z),x] = 0$$

where

$$F(x,y) = [\varphi(x),\psi(y)] + [\psi(x),\phi(y)].$$

Connection with CYBE

Known fact

$\varphi : L \to L$ is an R-matrix iff

$$[F(x,y),z] + [F(z,x),y] + [F(y,z),x] = 0 \qquad (\star)$$

where

$$F(x,y) = [\varphi(x),\varphi(y)] - \varphi([\varphi(x),y] + [x,\varphi(y)]).$$

F(x,y) = 0 is the (famous) QYBE.

Question

Study skew-symmetric solutions of $(\star)$ on various Lie algebras.

Symmetric solutions of $(\star)$ appear in the study of Lie-admissible power-associative algebras (Benkart 1984).

What happens when $HomLie(L)$ is a Jordan algebra

Theorem

Let L be a finite-dimensional Lie algebra over an algebraically closed field such that $\mathrm{HomLie}(L)$ forms a Jordan algebra. Then one of the following holds:

(*) $HomLie(K) \simeq K$.

(♢) $L = A \oplus B$ (as vector space), $A, B \neq 0$, $[[A, A], B] = 0$, $[[B, B], A] = 0$.

(♡) $0 \neq A \subseteq B \subset L$ (vector spaces), $\dim A + \dim B = \dim L$, $[[A, A], B] = 0$, $[[B, B], A] = 0$.

W_G in characteristic zero provides an infinite-dimensional counterexample to this theorem: neither of the conditions is satisfied.

$\diamondsuit$ and $\heartsuit$

Question

Study the properties $\diamondsuit$ and $\heartsuit$.

Conjecture 1

$\heartsuit$ has something to do with subalgebras of codimension 1.

Conjecture 2

(i) If L satisfies $\diamondsuit$, the $L/Rad(L)$ is the direct sum of $sl(2)$'s.

(ii) If L satisfies $\heartsuit$, then $L/Rad(L)$ is the direct sum of $sl(2)$'s and the Zassenhaus algebras.

That's all. Thank you.