SELECTED PROBLEMS IN MATHEMATICS
SYNOPSIS OF COURSE AT OU, SUMMER SEMESTER 2016

PASHA ZUSMANOVICH

Each lecture lasted 1.5 hours.

Lecture 1. Iterative correlation matrices

Problem 1.1. Does the negative entropy of a matrix A, defined as

$$-E(A) = \sum_{\lambda \in Spec(A)} \lambda \, ln(\lambda)$$

increase with iterative correlations?

If not (most probably), how we can modify it, or invent another function of matrix which will increase or decrease with iterative correlations?

Problem 1.2. Can we use some form of contraction mapping theorem to prove the convergence of iterative correlations?

Problem 1.3. Do iterative correlations of the 4×4 matrix

$$\begin{pmatrix} 1 & -0.73333333333 & -0.2 & 0.2 \\ -0.73333333333 & 1 & -0.2 & 0.2 \\ -0.2 & -0.2 & 1 & 0.33333333333 \\ 0.2 & 0.2 & 0.33333333333 & 1 \end{pmatrix}$$

converge? If not, what makes this matrix special? Are there other examples?

[1] C.-H. Chen, *Generalized association plots: information visualization via iterative generated correlation matrices*, Statistica Sinica **12** (2002), N1, 7–29.

[2] J.B. Kruskal, *A theorem about CONCOR*, Technical Report MH 2C-571, Bell Laboratories, 1978.

Lecture 2. Gravitational approach to clustering and alternative laws of gravitation

[1] E.G. Adelberger, B.R. Heckel, and A.E. Nelson, *Tests of the gravitational inverse-square law*, Ann. Rev. Nucl. Part. **53** (2003), 77–121; arXiv:hep-ph/0307284.

[2] A.K. Dewdney, *The Planiverse*, Springer, 2000.

[3] M. Milgrom, *General virial theorem for modified-gravity MOND*, Phys. Rev. D **89** (2014), 024016; arXiv:1311.2579.

Lecture 3. Composition of binary maps into ternary maps

Problem 3.1. Give a closed form formula, or recurrence formula for the numbers of ternary maps on an n-element set represented in the form $(x * y) * z$ and/or $x * (y * z)$ for some binary map $*$. Do the patterns one observes for small n remain? In particular, is it true that there is a relatively small number of "overlaps", i.e. these maps are mostly different for different $*$'s?

Problem 3.2. Compute further few terms of the number of commutative (= symmetric) ternary maps represented in the form $(x*y)*z$, and the number of such maps represented in such form for *commutative* $*$. Are these two numbers coincide?

The same questions may be posed when the maps are counted up to permutation.

Date: Last minor revision September 4, 2016.

[1] P. Zusmanovich, *On the last question of Stefan Banach*, Expos. Math., to appear; arXiv:1408.2982.

LECTURE 4. ALGEBRAIC SYSTEMS

General algebraic systems, groups, semigroups, rings, fields, vector spaces, algebras. Simple algebras. Isomorphism.

[1] R.D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, 1966; reprinted in a slightly corrected form by Dover, 1995.

LECTURE 5. LOW-DIMENSIONAL ALGEBRAS. LIE ALGEBRAS

Lie algebras. Any associative algebra A can be turned into a Lie algebra with respect to commutator $[a,b] = ab - ba$. Commutant $[L,L]$ of a Lie algebra L.

Claim 5.1. *Any* 1*-dimensional algebra over a field* K *is either trivial, or isomorphic to* K.

Theorem 5.2. *Any* 2*-dimensional Lie algebra is either abelian, or isomorphic to* $\langle x, y \,|\, [x,y] = x \rangle$.

[1] N. Jacobson, *Lie Algebras*, Interscience Publ., 1962; reprinted by Dover, 1979.

LECTURE 6. LOW-DIMENSIONAL LIE ALGEBRAS

Center of a Lie algebra. Direct sum of Lie algebras.

Examples of 3-dimensional Lie algebras: $sl_2(K)$ as commutant or quotient of $gl_2(K)$, simplicity of $sl_2(K)$. Heisenberg Lie algebra.

Sketch of classification of 3-dimensional Lie algebras.

Claim 6.1. *Over an algebraically closed field, every Lie algebra has a* 2*-dimensional subalgebra.*

LECTURE 7. LIE ALGEBRAS WITH GIVEN PROPERTIES OF SUBALGEBRAS. COMMUTATORS

Quaternions, $so_3(\mathbb{R})$.

Extension of the base field, forms.

Claim 7.1. *All proper subalgebras of* $so_3(\mathbb{R})$ *are* 1*-dimensional.*

Problem 7.1. Is it true that any finite-dimensional Lie algebra over any field, all whose proper subalgebras are 1-dimensional, has dimension ≤ 3?

Minimal nonabelian Lie algebras constructed from the vector space V and a linear map $f : V \to V$: they are of the form $V \oplus Kx$, where V is an abelian subalgebra, and $[v,x] = f(v)$ for $v \in V$.

Generators and relations. Tarski's monsters.

Problem 7.2. Do there exist infinite-dimensional Lie algebras, all whose proper subalgebras are 1-dimensional?

Claim 7.2. *If in a Lie algebra* L *every element of commutant is a commutator, and* $[L,L] \cap Z(L) \neq 0$, *then* L *contains a* 3*-dimensional Heisenberg Lie subalgebra.*

Claim 7.3. *In* $sl_2(K)$ *any element is a commutator.*

Sketch of the proof. As $B^{-1}[X,Y]B = [B^{-1}XB, B^{-1}YB]$ (what amounts to saying that $X \mapsto B^{-1}XB$ is an automorphism of $sl_2(K)$), it is enough to prove the statement for matrices in Jordan normal form. We have only two possibilities:

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

□

LECTURE 8. LIE ALGEBRAS – VARIOUS THINGS

Witt algebra as algebra of differential operators. $sl_2(K)$ as its subalgebra spanned by $\langle \frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx} \rangle$.
Proof of Claim 7.1:

Lemma 8.1. *Any 3-dimensional perfect algebra does not have 2-dimensional abelian subalgebras.*

Lemma 8.2. *Any Lie algebra of the form $D^{(-)}$, where D is associative division algebra over a field of characteristic zero, does not have 2-dimensional nonabelian subalgebras.*

Proof. Suppose we have for some $x, y \in D$, $xy - yx = x$. Then $[x, x^{-1}y] = 1$, what can be handled, for example, by passing to the extension of the base field and using trace argument. □

Structure constants of a Lie algebra. Problem to decide whether two Lie algebras are isomorphic reduces to solving quadratic systems.

Problem 8.1. Develop a program attempting to decide whether two given Lie algebras are isomorphic, by combining GAP capabilities with Gröbner bases calculations.

LECTURE 9. LIE ALGEBRAS – VARIOUS THINGS. II

In a free 3-step nilpotent Lie algebra of rank 4, generated by x, y, z, t, the element $[x,y] + [z,t]$ is not represented as a commutator.

Problem 9.1. Describe a class of Lie algebras for which every element of commutant is a commutator.

Lie p-algebras. Example of p-structures: abelian algebra, Lie algebra of the form $A^{(-)}$.
Lie derivation algebra of an algebra.

Claim 9.1. *For an arbitrary algebra A over a field of characteristic p, and $D \in \mathrm{Der}(A)$, $D^p \in \mathrm{Der}(A)$.*

LECTURE 10. DERIVATIONS, LIE p-ALGEBRAS

$\mathrm{Der}\, K[x] \simeq \mathrm{Witt}$.

Problem 10.1. Does any finite-dimensional Lie algebra can be represented as a derivation algebra of some (nonassociative) algebra?

Inner derivations, outer derivations. Outer derivations as the first cohomology group.

Problem 10.2 (very vague)**.** Compute cohomology of various Lie algebras.

Problem 10.3. Improve existing computer algorithms for computation of cohomology of Lie algebras.

p-structures on the 2-dimensional nonabelian Lie algebra.

Problem 10.4 (Jacobson)**.** Is it true that any (possibly infinite-dimensional) Lie algebra satisfying the condition $x^{[p]^{n(x)}} = x$ is abelian?

Lecture 11. Lie p-algebras, gradings

Theorem 11.1. *If a Lie p-algebra L satisfies $x^{[p]} = x$ for any $x \in L$, then L is abelian.*

Sketch of the proof. We have $\lambda^p = \lambda$ for any element λ of the base field, hence the base field coincides with $GF(p)$. As any $\mathrm{ad}\,x$ is annihilated by polynomial $f(t) = t^p - t$ with roots in the base field, then we have root space decomposition with respect to $\mathrm{ad}\,x$. If we have non-zero roots, then we have 2-dimensional nonabelian subalgebra which also satisfies the same property $x^{[p]} = x$, what is impossible by computation from the previous lecture. Hence $\mathrm{ad}\,x$ is nilpotent, what implies $\mathrm{ad}\,x = 0$. □

Relationship between derivations and automorphisms: if D is a derivation, then $exp(D)$, suitably defined, is an automorphism.

Gradings. Appearance of gradings as root space decomposition with respect to adjoint maps, derivations, automorphisms.

Example of non-semigroup grading from [1].

Problem 11.1. Is any grading on a finite-dimensional simple Lie algebra over a field of characteristic zero, a semigroup grading?

[1] A. Elduque, *More non-semigroup Lie gradings*, Lin. Algebra Appl. **431** (2009), N9, 1603–1606; arXiv:0809.4547.

Lecture 12. Number of semisimple Lie algebras. Operads

Semisimple Lie algebras in zero and positive characteristic. Example of a semisimple Lie algebra in characteristic p which is not the direct sum of simples ($\mathsf{sl}(2) \otimes K[t]/(t^p) + \frac{d}{dt}$).

Problem 12.1. Investigate the function $f(n)$, where $f(n)$ is the number of non-isomorphic finite-dimensional semisimple Lie algebras over an algebraically closed field of characteristic zero, of dimension n.

Operads. Koszul duality of binary quadratic operads. Tensor product of algebras of dual operads as Lie algebras.

Lecture 13. Operads

Examples of Koszul dual operads: (associative, associative), (Lie, associative commutative).

Problem 13.1. Find "interesting" Lie algebras represented as the tensor product of algebras over Koszul dual operads.

Poincaré series of an operad. Examples:

$$g_{Ass}(t) = \sum_{n=1}^{\infty} (-1)^n \frac{n!}{n!} t^n = -\frac{t}{1+t}$$

$$g_{Comm}(t) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} t^n = e^{-t} - 1$$

$$g_{Lie}(t) = \sum_{n=1}^{\infty} (-1)^n \frac{(n-1)!}{n!} = -\ln(1+t).$$

Ginzburg–Kapranov criterion for Koszulity of operads.

Problem 13.2. Is it true that for $n > 8$ the inverse power series of the polynomial $t - t^n + t^{2n-1}$ has non-negative coefficients?

[1] V. Dotsenko, M. Markl, and E. Remm, *Non-koszulness of operads and positivity of Poincaré series*, arXiv:1604.08580.

[2] A. Dzhumadil'daev and P. Zusmanovich, *The alternative operad is not Koszul*, Experiment. Math. **20** (2011), N2, 138–144; Corrigendum: **21** (2012), N4, 418; arXiv:0906.1272.

E-mail address: `pasha.zusmanovich@gmail.com`