# MEASURE THEORY AND INTEGRATION
# SYNOPSIS OF A COURSE AT OU, WINTER SEMESTERS 2016/2017 AND 2018/2019

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Each class lasted 3 (astronomical) hours (1.5 hours lecture + 1.5 hours tutorial).

## Lecture 1. Topic of the course. Refreshers on set theory and topology

Topic of the course. What is wrong with the Riemann integral? (According to [C, §1] and [K, §1.4]).

The stories of Lebesgue (difficulties with habilitation in 1902) and Grothendieck (invented measure theory himself).

Problem of measure. Inadequacy of Jordan measure. Banach-Tarski paradox. (According to [BK, §1], [T, §1.1]).

Refresher on basics of set theory. Operations on sets (union, intersection, difference, complement, symmetric difference). De Morgan's laws. Correspondence between set-theoretic operations and logical operation from the predicate calculus.

Venn diagrams.

**Exercise 1.1.** Whether it is true that intersection is distributive with respect to union, i.e. whether for any three sets $A$, $B$, $C$,

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

*Answer.* It is true. $\square$

Refresher on basics of topology. Definition of topological space. Importance of topological structures. Open sets, close sets, base, closure, interior. Trivial and discrete topology.

Euclidean topology on $\mathbb{R}$ defined as a topology with the base consisting of all open intervals.

**Exercise 1.2.** Prove that a closed interval is indeed a closed set in this topology.

**Exercise 1.3.** How many different topologies are there on the 2-element set?

*Answer.* 4:

$$\begin{aligned}
&\{\varnothing, \{1,2\}\}\\
&\{\varnothing, \{1\}, \{1,2\}\}\\
&\{\varnothing, \{2\}, \{1,2\}\}\\
&\{\varnothing, \{1\}, \{2\}, \{1,2\}\}.
\end{aligned}$$

$\square$

**Exercise 1.4.** Whether $\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q}$ are open or closed in the Euclidean (natural) topology on $\mathbb{R}$?

*Answer.* Both of them are neither open, nor closed. $\square$

*Date*: last modified December 18, 2018.

**Definition 2.1.** A ring of sets is a nonempty set of subsets of a universe $X$ closed with respect to all basic set operations (union, intersection, difference, symmetric difference).

**Fact 2.2.** *Any ring of sets $\Sigma$ contains $\varnothing$.*

*Proof.* Since $\Sigma$ is nonempty, it contains some set $A$, hence $\varnothing = A \setminus A \in \Sigma$. □

The minimal possible ring of sets is $\{\varnothing\}$, the maximal one is $P(X)$.

**Definition 2.3.** An algebra of sets is a ring of sets which contains the universe $X$.

**Fact 2.4.** *For any algebra of sets $\Sigma$, it holds $A \in \Sigma \Rightarrow X \setminus A \in \Sigma$.*

*Proof.* Follows from the fact that $X \in \Sigma$ and $\Sigma$ is closed with respect to difference. □

The minimal possible algebra of sets is $\{\varnothing, X\}$, the maximal one is also $P(X)$.

**Exercise 2.5.** Which of the following is a ring/algebra of sets:

(i) The set of all countable subsets of an infinite set $X$.
(ii) The set of all intervals in $\mathbb{R}$.

*Answer.* (i) It is a ring of sets, and it is an algebra of sets if and only if $X$ is countable.
(ii) It is not a ring of sets. □

**Theorem 2.6.** *Any ring of sets is a (usual) commutative ring with respect to operations $A \cdot B = A \cap B$ and $A + B = A \triangle B$.*

Note: the role of zero plays $\varnothing$.
In such a ring, it holds $A + A = 0$ (i.e., the ring has characteristic 2) and $A \cdot A = A$. Such rings are called *boolean rings*. Any boolean ring can be represented as a ring of sets (Stone's theorem), but we will not need this fact and it is out of the scope of this course.

Refresher about Galois fields and characteristic. $GF(2)$ is represented as an algebra of sets on any nonempty universe $X$ by assigning 0 to $\varnothing$, and 1 to $X$.

**Definition 2.7.** An algebra of sets which is closed with respect to *countable* intersections and unions, is called $\sigma$-algebra. The pair $(X, \Sigma)$, where $\Sigma$ is a $\sigma$-algebra on the set $X$, is called a *measurable space*.

**Exercise 2.8.** Give an example of a $\sigma$-algebra which is also a topology.

*Answer.* 1) Trivial topology $\{\varnothing, X\}$ and discrete topology $P(X)$.
2) $\{\varnothing, A, X \setminus A, X\}$, assuming $X$ contains more then 1 element, and $A$ is a proper subset of $X$. □

**Lemma 2.9.** *Intersection of any family of $\sigma$-algebras is a $\sigma$-algebra.*

**Definition 2.10.** Let $T$ be a set of subsets of a set $X$. The smallest $\sigma$-algebra containing $T$ is called the $\sigma$-algebra generated by $T$ and is denoted by $\langle T \rangle$.

*Proof that such $\sigma$-algebra always exists.* Consider the family of all $\sigma$-algebras containing $T$. This family contains $P(X)$, hence is nonempty. $\langle T \rangle$ is the intersection of all $\sigma$-algebras in this family. □

**Definition 2.11.** A Borel $\sigma$-algebra of a topological space $X$, denoted by $\mathscr{B}(X)$, is the $\sigma$-algebra generated by the topology (i.e., the set of open subsets) of $X$.

An important role in what follows will be played by the Borel $\sigma$-algebra $\mathscr{B}(\mathbb{R})$ of $\mathbb{R}$ with respect to the Euclidean (natural) topology, or, more general, by $\mathscr{B}(\mathbb{R}^n)$.

Descriptive set theory. Construction of $\mathscr{B}(\mathbb{R})$ via transfinite induction. (Sketchy, without proofs, according to [T, pp. 87–88]).

**Exercise 3.1.** Find the $\sigma$-algebra generated by the following sets of subsets of a nonempty set $X$:

(i) $\{A\}$, $A \subset X$;
(ii) $P(A)$, $A \subset X$;
(iii) all 2-element subsets of $X$;
(iv) (Assuming $X = \mathbb{R}$)
  (a) $\{(-\infty, a) \mid a \in \mathbb{R}\}$;
  (b) $\{(-\infty, a] \mid a \in \mathbb{R}\}$;
  (c) $\{(-\infty, a) \mid a \in \mathbb{Q}\}$;
  (d) $\{(-\infty, a] \mid a \in \mathbb{Q}\}$.

*Answer.* (i) $\{\varnothing, A, X \backslash A, X\}$ if $A$ is proper subset; $\{\varnothing, X\}$ otherwise.
(ii) ?
(iii) All countable subsets of $X$ and all complements to countable subsets of $X$.
(iv) $\mathscr{B}(\mathbb{R})$ in all the cases. □

**Definition 3.2.** Let $(X,\Sigma)$ and $(Y,\Omega)$ be two measurable spaces. Their product is defined as a measurable space $(X \times Y, \langle \Sigma \times \Omega \rangle)$, where $\Sigma \times \Omega$ is the set of all sets of the form $A \times B$, where $A \in \Sigma$, $B \in \Omega$. The $\sigma$-algebra $\langle \Sigma \times \Omega \rangle$ is called the product of $\sigma$-algebras $\Sigma$ and $\Omega$.

This definition readily generalizes to the case of several measurable spaces. In a particular case of the product of the measurable space with itself, we will speak about the *power* of a measurable space (or the power of a $\sigma$-algebra).

**Theorem 3.3.** $\langle \mathscr{B}(\mathbb{R})^n \rangle = \mathscr{B}(\mathbb{R}^n)$*, where n is a nonnegative integer or infinity.*

(In other words, for $\mathbb{R}$, the operations of raising to the (Cartesian) power and taking the Borel $\sigma$-algebra commute).

*Proof* (after [BK, pp. 12–13]). As we want to prove the equality of two $\sigma$-algebras generated by some sets, it is enough to prove that any element of the generating set of the first (respectively, second) $\sigma$-algebra belongs to the second (respectively, first) one.

Any open set $O$ in $\mathbb{R}^n$ can be represented as a countable union of ($n$-dimensional) rectangles in $\mathbb{R}^n$. Every rectangle in $\mathbb{R}^n$ is a product of intervals in $\mathbb{R}$, and hence belongs to $\mathscr{B}(\mathbb{R})^n$. Hence $O$ belongs to $\langle \mathscr{B}(\mathbb{R})^n \rangle$. Since $\mathscr{B}(\mathbb{R}^n)$ is generated by open subsets in $\mathbb{R}^n$, this shows that $\mathscr{B}(\mathbb{R}^n) \subseteq \langle \mathscr{B}(\mathbb{R})^n \rangle$.

Conversely, let $A_1, \dots, A_n \in \mathscr{B}(\mathbb{R})$, so $A_1 \times \dots \times A_n \in \mathscr{B}(\mathbb{R})^n$. We have

$$A_1 \times \dots \times A_n = (A_1 \times \mathbb{R} \times \dots \times \mathbb{R}) \cap (\mathbb{R} \times A_2 \times \dots \times \mathbb{R}) \cap \dots \cap (\mathbb{R} \times \mathbb{R} \times \dots \times A_n). \tag{1}$$

The sets of the form $O \times \mathbb{R} \times \dots \times \mathbb{R}$, where $O$ is an open subset of $\mathbb{R}$, are open in $\mathbb{R}^n$, and hence belong to $\mathscr{B}(\mathbb{R}^n)$. The closure of these sets with respect to operations of a $\sigma$-algebra (countable unions and intersections, and complement) is $\mathscr{B}(\mathbb{R}) \times \{\mathbb{R}\} \times \dots \times \{\mathbb{R}\}$, so the latter set lies in $\mathscr{B}(\mathbb{R}^n)$, i.e. the first set in the intersection at the right-hand side of (1) belongs to $\mathscr{B}(\mathbb{R}^n)$. Similarly for the rest of the sets in this intersection, what implies $A_1 \times \dots \times A_n \in \mathscr{B}(\mathbb{R}^n)$, and hence $\langle \mathscr{B}(\mathbb{R})^n \rangle \subseteq \mathscr{B}(\mathbb{R}^n)$. □

**Definition 3.4.** Let $(X,\Sigma)$ and $(Y,\Omega)$ be two measurable spaces. A map $f: X \to Y$ is called measurable, if for any $B \in \Omega$, $f^{-1}(B) \in \Sigma$.

**Example 3.5.** The projection $X \times Y \to X, (x,y) \mapsto x$ is measurable.

**Lemma 3.6** ([BK, Proposition 2.4])**.** *Let* $(X,\Sigma)$ *and* $(Y,\Omega)$ *be two $\sigma$-algebras, and* $\Omega$ *is generated by (a set of subsets of Y) T. Then the map* $f: X \to Y$ *is measurable, if for any* $B \in T$, $f^{-1}(B) \in \Sigma$.

*Proof.* Let $\Omega' = \{B \in \Omega \mid f^{-1}(B) \in \Sigma\}$. It is straightforward to verify that $\Omega'$ is a $\sigma$-algebra. We have $T \subseteq \Omega' \subseteq \Omega$. But since $\Omega$ is generated by $T$, we have $\Omega' = \Omega$, and $f$ is measurable. $\square$

**Theorem 3.7** ("A continous map is measurable")**.** *Any continuous map between two topological spaces is measurable with respect to the corresponding Borel $\sigma$-algebras.*

*Proof* (after [BK, Example 2 on p. 10]). For continuous map, the preimage of an open set is open and hence belongs to the $\sigma$-algebra of the domain. Since open sets generated this $\sigma$-algebra, the statement follows from Lemma 3.6. $\square$

## Lecture 4. The extended real line. Measure. Further properties of measurable functions

4.1. **The extended real line (according to** [BK, Chap. 2, Example on p. 13]**).** Consider the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. We turn it into a measurable space by declaring a set $A \subseteq \overline{\mathbb{R}}$ to be measurable if and only if $A \cap \mathbb{R}$ is measurable (with respect to Borel measure). (In other words, we are taking all measurable subsets of $\mathbb{R}$ and possibly adding to them $+\infty$, or $-\infty$, or both). It is easy to see that the set of all such subsets of $\overline{\mathbb{R}}$ forms a $\sigma$-algebra. Occasionally, we will need to consider the measurable space $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{+\infty\}$ consisting of the nonnegative part of $\overline{\mathbb{R}}$ (or, what is the same, adding to the measurable space $\mathbb{R}_{\geq 0}$ one element $+\infty$).

4.2. **Additive functions.**

**Definition 4.1.** A function $f$ is called *additive non-negative function* if it is defined on an algebra of sets $\Sigma$ with values in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$, and $f(A \cup B) = f(A) + f(B)$ for any $A, B \in \Sigma$ such that $A \cap B = \varnothing$, and $f(\varnothing) = 0$.

**Example 4.2.** A function on any algebra of sets sending everything to zero.

**Example 4.3.** Take $\Sigma$ as the 4-element algebra in Exercise 2.8 2), and define

$$\varnothing \mapsto 0, \quad A \mapsto \alpha, \quad X \backslash A \mapsto \beta, \quad X \mapsto \alpha + \beta$$

for any two nonnegative numbers $\alpha, \beta$.

**Lemma 4.4.** *Let $f$ be an additive nonnegative function on an algebra of sets $\Sigma$, and $A \subseteq B$, $A, B \in \Sigma$. Then:*

(i) $f(B \backslash A) = f(B) - f(A)$;
(ii) $f(A) \leq f(B)$.

*Proof.* (i) $f(B) = f((B \backslash A) \cup A) = f(B \backslash A) + f(A)$.
(ii) Follows from (i), as $f(B \backslash A) \geq 0$. $\square$

**Theorem 4.5** (Inclusion-exclusion principle)**.** *Let $f$ be an additive non-negative function on an algebra of sets $\Sigma$. Then*

$$f(A \cup B) = f(A) + f(B) - f(A \cap B)$$

*for any $A, B \in \Sigma$, and*

$$f(A \cup B \cup C) = f(A) + f(B) + f(C) - f(A \cap B) - f(A \cap C) - f(B \cap C) + f(A \cap B \cap C)$$

*for any $A, B, C \in \Sigma$.*

Generalization of this to the case of $n$ sets.

## 4.3. **Measure.**

**Definition 4.6.** Let $(X,\Sigma)$ be a measurable space. A *measure* on $(X,\Sigma)$ is a nonnegative additive map $\mu$ on $\Sigma$ which is countably additive, i.e. for any countable set $\{A_i\}_{i=1}^{\infty}$, $A_i \in \Sigma$, it holds

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

The triple $(X,\Sigma,\mu)$ is called a *measure space*.

**Example 4.7.** Counting measure:

$$\mu(A) = \begin{cases} |A|, & \text{if } A \text{ is finite;} \\ +\infty, & \text{if } A \text{ is infinite .} \end{cases}$$

**Example 4.8.** Dirac measure. Let $X$ be a set, $x \in X$, and $a \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$.

$$\mu(A) = \begin{cases} a, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Later we will deal with the Borel measure, and its extension, the Lebesgue measure, the central subject of our course. The Borel measure is defined on $\mathscr{B}(\mathbb{R}^n)$ as the extension of the function $\mu$ defined on $n$-dimensional rectangles:

$$\mu([a_1,b_1] \times \cdots \times [a_n,b_n]) = (b_1 - a_1)\dots(b_n - a_n). \tag{2}$$

The existence of such measure is not obvious at all and will be established later. Intuitively, it is obtained by propagation of the function (2) by countable additivity from $n$-dimensional rectangles to the whole transfinite hierarchy of Borel sets.

Probability space is, essentially, the same as a finite measure space (i.e., measure space in which the measure of every set is finite) with a probability measure normalized by the condition $\mu(X) = 1$. Taking different probability distributions $f$ (uniform, normal, Poisson, etc.), we may get different examples of measures defined by

$$\mu(A) = \int_A f(t)\,\mathrm{d}t.$$

Integration here is performed in the sense of Lebesgue, what will be defined later.

## 4.4. **Further properties of measurable functions.** (had to be in/after Lecture 3)

**Lemma 4.9.** inf *and* sup*, considered as functions* $\overline{\mathbb{R}}^{\infty} \to \overline{\mathbb{R}}$*, are measurable functions.*

(For the proof, see [BK, p. 13]).

**Definition 4.10.** A *characteristic function* of a subset $A$ of the universe $X$ is the function $\mathbf{1}_A : X \to \mathbb{R}$ defined as

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

# Lecture 5. Properties of measurable functions. Monotonicity principle

**Lemma 5.1.** *If $X$ is a measurable space, then $\mathbf{1}_A$ is measurable iff $A$ is measurable.*

**Theorem 5.2.** *The set of measurable functions from an arbitrary measurable space to $\mathbb{R}$ forms a vector space.*

(In other words, these measurable functions are closed with respect to linear combinations. For the proof, see [BK, p. 14]).

Similarly, if we restrict ourselves to linear combinations with *nonnegative* coefficients (to avoid undefined sum $-\infty + \infty$), then the set of all measurable functions from a measurable space to $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ is closed with respect to such combinations.

**Lemma 5.3.** *If $f_1, f_2, \dots$ are measurable functions from a measurable space $S$ to $\overline{\mathbb{R}}$, then $\sup_n f_n$ and $\inf_n f_n$ are measurable.*

*Proof.* (after [BK, p. 15]) We have $\sup_n f_n = \sup \circ (f_1, f_2, \dots)$, where

$$(f_1, f_2, \dots) : S \to \overline{\mathbb{R}}^\infty$$
$$x \mapsto (f_1(x), f_2(x), \dots).$$

Let us prove that $(f_1, f_2, \dots)$ is measurable. Let $X_1, X_2, \dots$ be measurable subsets of $\overline{\mathbb{R}}$. Then

$$(f_1, f_2, \dots)^{-1}(X_1 \times X_2 \times \dots) = \{x \in S \mid (f_1, f_2, \dots)(x) \in X_1 \times X_2 \times \dots\}$$
$$= \{x \in S \mid f_i(x) \in X_i, i = 1, 2, \dots\} = \bigcap_{i=1}^{\infty} f_i^{-1}(X_i).$$

Since each $f_i$ is measurable, $f_i^{-1}(X_i)$ is measurable, and hence $\bigcap_{i=1}^{\infty} f_i^{-1}(X_i)$ is measurable. Since the sets of the form $X_1 \times X_2 \times \dots$, where each $X_i$ is measurable, generate the (Borel) $\sigma$-algebra on $\overline{\mathbb{R}}^\infty$, by Lemma 3.6 $(f_1, f_2, \dots)$ is measurable. By Fact 4.9, sup is a measurable function, and $\sup_n f_n$, being a composition of measurable functions, is measurable.

Similarly for inf. □

Monotonicity principle ([BK, Proposition 2.8]).

**Theorem 5.4** ("Countable additivity is equivalent to continuity")**.** *For a nonnegative additive function $\mu$ on a $\sigma$-algebra $\Sigma$, the following are equivalent:*

(i) *$\mu$ is a measure;*

(ii) *$\mu$ is continuous from below, i.e. for any non-decreasing countable chain of sets $A_1 \subseteq A_2 \subseteq \dots$, $A_i \in \Sigma$, it holds $\mu(A_n) \to \mu(\bigcup_{i=1}^{\infty} A_i)$ as $n \to \infty$.*

(iii) *$\mu$ is continuous from above, i.e. for any non-increasing countable chain of sets $A_1 \supseteq A_2 \supseteq \dots$, $A_i \in \Sigma$ such that $\mu(A_n) < +\infty$ for some $n$, it holds $\mu(A_n) \to \mu(\bigcap_{i=1}^{\infty} A_i)$ as $n \to \infty$.*

*Proof.* For the proof of (i) ⇔ (ii) see, e.g., [MP, Theorem 1.3.3]. The main idea is to transform a non-decreasing chain $\{A_i\}$ to a family of pairwise disjoint sets $\{B_i\}$ via $B_1 = A_1$, and $B_i = A_i \setminus A_{i-1}$ for $i > 1$. To prove (i) ⇔ (iii) is Homework 9. □

The condition that $\mu(A_n) < +\infty$ for some $n$ in heading (iii) (essentially, $\mu(A_n) < +\infty$ for *all* $n$, as we may throw away any finite number of $A_i$'s) is important. Indeed, in the decreasing chain of intervals $(n, +\infty)$, $n = 1, 2, \dots$, each interval has measure $+\infty$, and their intersection is $\varnothing$, i.e. of measure zero.

## Lecture 6. Existence of Borel measure

Outer measure, Caratheodory measure (measure on the $\sigma$-algebra of measurable sets with respect to an outer measure), extension of measure from the generating set to the whole $\sigma$-algebra (according to [BK, Chap. 11, pp. 118–123]).

## Lecture 7. Uniqueness of Borel measure, completion of a measure

**Theorem 7.1** (Uniqueness of a measure, [BK, Proposition 7.1])**.** *Let $\mu, \nu$ be two measures defined on a measurable space $(S, \Sigma)$, $\Sigma$ is generated by a $\cap$-closed set $\mathcal{E}$, and $\mu, \nu$ coincide on any element of $\mathcal{E}$. If there is a countable set $\{E_n\}_{n=1}^{\infty}$ of elements of $\mathcal{E}$ such that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$, $\bigcup_{n=1}^{\infty} E_n = S$, and $\mu(E_n) = \nu(E_n) < \infty$ for any $n = 1, 2, \dots$, then $\mu = \nu$.*

**Definition 7.2.** A set $\mathcal{D}$ of subsets of a universe $S$ is called a *D-system*[†] if the following conditions are satisfied:

(i) $S \in \mathcal{D}$;

[†] *Dynkin system* in [BK].

(ii) $\mathcal{D}$ is closed with respect to complements;
(iii) If $\{A_i\}_{i=1}^{\infty}$ is a countable set of pairwise disjoint members of $\mathcal{D}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{D}$.

Similarly to the situation with (a stronger notion of) $\sigma$-algebras (Definition 2.10), we have the notion of a D-system generated by a set of subset of the universe $S$. Its existence is proved in the same way as in the case of $\sigma$-algebras. First we establish

**Lemma 7.3.** *Intersection of any family of D-systems is a D-system.*

This is a direct analog of Lemma 2.9. Now, for any set $T$ of subsets of $S$, the D-system generated by $T$ is the intersection of all D-systems containing $T$. Its existence is guaranteed by the previous lemma.

**Lemma 7.4.** † *A D-system generated by a $\cap$-closed set of subsets of the universe, is a $\sigma$-algebra.*

Then the proof Theorem 7.1 is conducted as in [BK, Chap. 7, p. 65].

**Corollary 7.5.** *The Borel measure is unique.*

*Proof.* See [BK, Example on p. 64]. □

Examples showing that the condition $\mu(E_n) = \nu(E_n) < +\infty$ in Theorem 7.1 is necessary.

**Proposition 7.6.** *Let $(S, \Sigma, \mu)$ be a measure space, $A \subset X$. Define a new function $\mu_A : \Sigma \to \overline{\mathbb{R}}_{\geq 0}$ by*

$$\mu_A(X) = \begin{cases} \mu(X), & \text{if } X \subseteq A \\ +\infty, & \text{otherwise.} \end{cases}$$

*Then $\mu_A$ is a measure.*

Now in the setup of Theorem 7.1, set $S = \mathbb{R}$, $\Sigma$ the Borel $\sigma$-algebra, $\mathcal{E}$ is the set of open intervals, and $E_n = (-n, n)$. Consider two measures: $\mu_\varnothing$ is an "infinity measure" taking value 0 on the empty set and $+\infty$ on all other sets, and $\mu_{(\frac{1}{2},\frac{1}{2})}$. Then all the conditions of Theorem 7.1 except of $\mu_\varnothing(E_n) = \mu_{(\frac{1}{2},\frac{1}{2})}(E_n) < +\infty$ are satisfied, and $\mu_\varnothing \neq \mu_{(\frac{1}{2},\frac{1}{2})}$.

Another example: Borel measure $\mu$ and counting measure $\nu$, i.e.

$$\nu(A) = \begin{cases} |A|, & \text{if } A \text{ is finite} \\ +\infty, & \text{if } A \text{ is infinite,} \end{cases}$$

$E_n = (-\infty, n)$, $n = 1, 2, \ldots$, and $\mathcal{E} = \{(-\infty, a) \mid a \in \mathbb{R}\}$.

**Definition 7.7.** For a measure space $(S, \Sigma, \mu)$, define a new $\sigma$-algebra $\widehat{\Sigma}$ on $S$ in one of the following equivalent ways:

(i) $\widehat{\Sigma}$ is a $\sigma$-algebra generated by $\Sigma \cup \{A \subseteq S \mid A \subseteq B$ for some $B \in \Sigma$ such that $\mu(B) = 0\}$
(ii) ~~$\widehat{\Sigma} = \{A \subseteq S \mid B \subseteq A \subseteq C$ for some $B, C \in \Sigma$ such that $\mu(B) = \mu(C)\}$~~
$\widehat{\Sigma} = \{A \subseteq S \mid B \subseteq A \subseteq C$ for some $B, C \in \Sigma$ such that $\mu(C \backslash B) = 0\}$

The measure $\widehat{\mu}$ on $\Sigma$ is defined, according to the definition (ii) as $\widehat{\mu}(A) = \mu(B) = \mu(C)$. The measure space $(S, \widehat{\Sigma}, \widehat{\mu})$ is called the *completion* of $(S, \Sigma, \mu)$.

*Proof of correctness of this definition (*after [BK, p. 23]*).* First we prove in a straightforward way that $\widehat{\Sigma}$ as in (ii) is indeed a $\sigma$-algebra. Now let $A \in \widehat{\Sigma}$ as in (ii). For the appropriate $B, C$ we may write $A = A \cup (A \backslash B)$, $A \backslash B \subseteq C \backslash B$, and $\mu(C \backslash B) = 0$, hence $A \in \widehat{\Sigma}$ as in (i). But the generating set in (i) obviously lies in $\widehat{\Sigma}$ defined as in (ii), so $\widehat{\Sigma}$'s in (i) and (ii) coincide. □

**Definition 7.8.** The *Lebesgue measure* on $\mathcal{B}(\mathbb{R}^n)$ is the completion of the Borel measure.

† Equivalent to [BK, Proposition 7.2].

## Lecture 8. Properties of Borel measure

Rules for manipulating the extended positive part of the real line $\mathbb{R}_{\geq 0} \cup \{+\infty\}$: $(+\infty)\cdot 0 = 0 \cdot (+\infty) = 0$. This is justified by the following: if we have the countable number of pairwise disjoint sets of measure zero (say, points on the real line), then, by countable additivity, their union is also of measure zero. This amounts to saying that zero, summed up infinite number of times, or $0\cdot(+\infty)$, is zero (had to be in Lecture 4).

Properties of Borel and Lebesgue measures (after [BK, pp. 24–26]).

**Theorem 8.1** (Vitali, [BK, Ex. 3.4])**.** *There are subsets of $\mathbb{R}$ not measurable with respect to Borel measure.*

(Or, in other words, $\mathscr{B}(\mathbb{R}) \neq P(\mathbb{R})$).

*Proof.* Consider the quotient of abelian groups $\mathbb{R}/\mathbb{Q}$. For any equivalence class in this quotient, pick one representative. Obviously, we can pick representatives lying in an arbitrary given interval, so let us pick representatives lying in $[0,1]$, and denote the set of all such representatives by $N$.

Let $A = \bigcup_{r\in\mathbb{Q}\cap[-1,1]}(N+r)$. Let $0 \leq x \leq 1$. Then $x = n + r$ for some $n \in N$ and $r \in \mathbb{Q}$, so $r = x - n$, and hence $0-1 \leq r \leq 1-0$. This shows that

$$[0,1] \subseteq A. \tag{3}$$

On the other hand, an arbitrary element in $N+r$ with $r \in [-1,1]$ lies between $0-1$ and $1+1$. This shows that

$$A \subseteq [-1,2]. \tag{4}$$

Let $x \in (N+r)\cap(N+s)$ for $r,s \in \mathbb{Q}$. Then $x = n + r = m + s$ for some $n,m \in N$ and $r,s \in \mathbb{Q}$, hence $n - m = s - r \in \mathbb{Q}$, and hence, by choice of $N$, $n = m$ and then $r = s$. This shows that the sets $(N+r)$, where $r \in \mathbb{Q}$, are pairwise disjoint.

Now assume that $N$ is measurable. By invariance of Borel measure with respect to translation, any set $N+r$ is measurable, and $\mu(N+r) = \mu(N)$. By countable additivity, $A$ is measurable, and $\mu(A) = \mu(N)\cdot\infty$, so $\mu(A)$ is equal to 0 or $\infty$, depending whether $\mu(N) = 0$ or $\mu(N) > 0$. But by (3) and (4), $1 \leq \mu(A) \leq 3$, a contradiction. $\square$

## Lecture 9. Lebesgue integral of nonnegative functions

Definition of elementary function and of Lebesgue integral for nonnegative functions according to [BK, Chap. 4, pp. 29-30]). The only difference is that we define an elementary function as a function assuming *countable* (and not finite, as in [BK]) number of values. This allows to have all the proofs from [BK] almost verbatim, while avoiding some complications like in the proof of Proposition 4.5. This is also more nice conceptually, as it resembles the definition of a discrete random variable.

"Markov's inequality" ([BK, Proposition 4.1]), properties of Lebesgue integral ([BK, Proposition 4.2]), Monotone Convergence Theorem ([BK, Proposition 4.3]).

Refresher on $\liminf$.

**Definition 9.1.** For a sequence of real numbers $\{x_n\}$, $\liminf_{n\to\infty} x_n$ is defined in one of the following equivalent ways:

(i) $\lim_{n\to\infty}(\inf_{m\geq n} x_m)$;

(ii) $\sup_n(\inf_{m\geq n} x_m)$;

(iii) $\inf_{\{x_{n_k}\}\text{is a convergent subsequence of }\{x_n\}}(\lim_{k\to\infty} x_{n_k})$.

Since the sequence $\{\inf_{m\geq n} x_m\}_{n=1}^{\infty}$ is non-decreasing, the limit in (i) always exists and coincides with (ii). For equivalence with (iii), see your favorite book on mathematical analysis.

"Fatou's lemma" ([BK, Proposition 4.4]).

**Theorem 9.2** (after [BK, Proposition 4.5]). *Let $f$ be a nonnegative elementary on a measure space $S$. Then*

$$\int f\,\mathrm{d}\mu = \sum_y y\cdot\mu(f=y).$$

*Proof.* Let $0\le h\le f$ be another elementary function. We have $S=\bigcup_y\{f=y\}$, and hence

$$\{h=z\}=\Big(\bigcup_y\{f=y\}\Big)\bigcap\{h=z\}=\bigcup_y\Big(\{f=y\}\bigcap\{h=z\}\Big)=\bigcup_y\{f=y,h=z\}.$$

By countable additivity of measure,

(5)
$$\mu(h=z)=\sum_y\mu(f=y,h=z),$$

and

$$\sum_z z\cdot\mu(h=z)=\sum_z\sum_y z\cdot\mu(h=z,f=y).$$

Since $h\le f$, in the last sum we have always $z\le y$, and so it is less than or equal to

(6)
$$\sum_z\sum_y y\cdot\mu(h=z,f=y).$$

Similarly with (5), we have $\mu(f=y)=\sum_z\mu(f=y,h=z)$, and hence the sum (6) is equal to

$$\sum_y y\cdot\mu(f=y).$$

To summarize: for any elementary function $0\le h\le f$ we have

$$\sum_z z\cdot\mu(h=z)\le\sum_y y\cdot\mu(f=y).$$

Since $\int f\,\mathrm{d}\mu$ is the supremum over all sums at the left-hand side of this inequality, we have

$$\int f\,\mathrm{d}\mu\le\sum_y y\cdot\mu(f=y).$$

On the other hand, since $f$ is elementary itself, we have the reverse inequality, and the statement is proved for any elementary $f$.

□

## LECTURE 10. PROPERTIES OF LEBESGUE INTEGRAL OF NONNEGATIVE FUNCTIONS (CONTINUATION)

Linearity of integral ([BK, Proposition 4.6]).

**Theorem 10.1** (Countable additivity of Lebesgue integral, [BK, Proposition 4.7]). *For any measurable functions $f_n\ge 0$, we have*

$$\int\Big(\sum_{n=1}^{\infty}f_n\Big)\mathrm{d}\mu=\sum_{n=1}^{\infty}\Big(\int f_n\,\mathrm{d}\mu\Big).$$

*Proof.* Let $g_k=\sum_{n=1}^k f_n$. Applying the Monotone Convergence Theorem to the sequence of measurable functions $0\le g_1\le g_2\le\dots$, and the linearity of integral, we get

$$\int\Big(\sum_{n=1}^{\infty}f_n\Big)\mathrm{d}\mu=\int\sup_k\Big(\sum_{n=1}^{k}f_n\Big)\mathrm{d}\mu=\lim_{k\to\infty}\Big(\int\Big(\sum_{n=1}^{k}f_n\Big)\mathrm{d}\mu\Big)=\lim_{k\to\infty}\Big(\sum_{n=1}^{k}\int f_n\,\mathrm{d}\mu\Big)=\sum_{n=1}^{\infty}\Big(\int f_n\,\mathrm{d}\mu\Big).$$

□

**Definition 10.2** (Integration over a domain). Let $f$ be a measurable nonnegative function on a measure space $S$, and $A$ a measurable subset of $S$. Then $\int_A f\,\mathrm{d}\mu$ is defined in one of the following equivalent ways:

(i) By restricting everything to $A$, i.e. as $\int f|_A\,\mathrm{d}\mu|_A$;

(ii) As $\int (f \cdot \mathbf{1}_A)\,\mathrm{d}\mu$.

**Theorem 10.3** ([BK, Proposition 4.8]). *Let $f \geq 0$ be measurable. Then*

$$\int f\,\mathrm{d}\mu = \int_0^\infty \mu(f > t)\,\mathrm{d}t.$$

(The integral at the right-hand side is understood with respect to Lebesgue measure on $\mathbb{R}$. Later, when we will discuss the relation of Lebesgue and Riemann integrals, we will see that it also can be interpreted as the Riemann integral).

*Proof.* Let

$$f_n = \frac{1}{2^n}\sum_{k=1}^{\infty} \mathbf{1}_{\{f > \frac{k}{2^n}\}}.$$

In other words, for any $x$ from the domain, $f_n(x)$ is the best approximation of $f(x)$ by dyadic rational numbers of the form $\frac{k}{2^n}$, so $\sup_n f_n = f$. Obviously, the bigger is $n$, the better is the approximation, so we have $0 \leq f_1 \leq f_2 \leq \ldots$. By Theorem 10.1,

$$\int f_n\,\mathrm{d}\mu = \frac{1}{2^n}\sum_{k=1}^{\infty}\int \mathbf{1}_{\{f > \frac{k}{2^n}\}}\,\mathrm{d}\mu = \frac{1}{2^n}\sum_{k=1}^{\infty}\mu(f > \frac{k}{2^n}).$$

Further, we have $\mu(f > \frac{k}{2^n}) = \mu(f_n > t)$ for $\frac{k}{2^n} < t \leq \frac{k+1}{2^n}$, so by Theorem 9.2,

$$\int_0^\infty \mu(f_n > t)\,\mathrm{d}t = \sum_{k=1}^{\infty}\mu(f > \frac{k}{2^n})\delta([\frac{k}{2^n}, \frac{k+1}{2^n}]) = \frac{1}{2^n}\sum_{k=1}^{\infty}\mu(f > \frac{k}{2^n})$$

(here $\delta$ is the Lebesgue measure on $\mathbb{R}$).

Combining the last two equalities, we get

$$\int f_n\,\mathrm{d}\mu = \int_0^\infty \mu(f_n > t)\,\mathrm{d}t. \tag{7}$$

Since $\{f_n\}$ is non-decreasing, for any given $t$ the sequence of functions $\{\mu(f_n > t)\}$ is non-decreasing, and $\sup_n \mu(f_n > t) = \mu(f > t)$. Passing in (7) to $\lim_{n\to\infty}$ and using the Monotone Convergence Theorem, we get the desired result. □

## Lecture 11. Properties of Lebesgue integral of nonnegative function (end). Lebesgue integral of arbitrary functions

**Exercise 11.1.** Let $(X,\Sigma),(Y,\Omega)$ be measurable spaces, $\mu$ a measure on $X$, and $\varphi : X \to Y$ a measurable map. Then the map $\mu \circ \varphi^{-1} : \Omega \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ assigning to any measurable subset $B$ of $Y$, the set $\mu(\varphi^{-1}(B))$, is a measure on $Y$.

**Theorem 11.2** (Change of variables, [BK, Proposition 4.9]). *Let $X, Y$ be measurable spaces, $f : Y \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ a measurable function, and $\varphi : X \to Y$ a measurable map. Then*

$$\int f\,\mathrm{d}(\mu \circ \varphi^{-1}) = \int (f \circ \varphi)\,\mathrm{d}\mu.$$

For the proof, utilizing the monotonicity principle, see [BK].

Lebesgue integral of an arbitrary measurable function, integrability, monotonicity and linearity of integral, Dominated Convergence Theorem ([BK, Chap. 5, pp. 41–43]).

Definition of Lebesgue integral over a measurable set.

**Theorem 11.3.**

(i) *Let $f$ be an integrable function, $A$ and $B$ two disjoint measurable sets. Then*

$$\int_{A\cup B} f\,\mathrm{d}\mu = \int_A f\,\mathrm{d}\mu + \int_B f\,\mathrm{d}\mu.$$

(ii) *If $A$ is a set of measure zero, then*

$$\int_A f \, \mathrm{d}\mu = 0.$$

## Lecture 12. Computation of various Lebesgue integrals

Unless stated otherwise, the integrals are taken with respect to the Lebesgue measure on $\mathbb{R}$.

**Exercise 12.1.** Compute $\displaystyle\int_{\mathbb{Q}} e^{-x^2} \, \mathrm{d}x$.

*Answer.* 0, as integral over any set of measure zero is zero. □

**Exercise 12.2.** Compute $\displaystyle\int_0^1 f(x) \, \mathrm{d}x$, where

$$f(x) = \begin{cases} x^3, & \text{if } x \notin \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{Q}. \end{cases}$$

*Answer.* When integrating, we can ignore any set of measure zero, so this integral is equal to $\displaystyle\int_0^1 x^3 \, \mathrm{d}x$. As the Lebesgue integral is equal to the Riemann integral, if the latter exists, we can evaluate it in the usual way, and it is equal to $\frac{1}{4}$. □

**Exercise 12.3.** Compute $\displaystyle\int_0^1 f(x) \, \mathrm{d}x$, where

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathcal{C} \\ \frac{1}{2^n}, & \text{if } x \text{ belongs to the interval thrown away in the process of construction of the } \mathcal{C} \text{ of length } \frac{1}{3^n}. \end{cases}$$

Here $\mathcal{C}$ is the Cantor set (for a nice and rigorous treatment of the Cantor set see, e.g., [KS, §3.2.2]; for a non-rigorous and visual treatment, see Wikipedia).

*Answer.* Since the Cantor set has measure zero, we may ignore it. There are $2^{n-1}$ intervals of the length $\frac{1}{3^n}$, so by Theorem 9.2, the integral is equal to

$$\sum_{n=1}^{\infty} \frac{1}{2^n} 2^{n-1} \frac{1}{3^n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{4}.$$

□

**Exercise 12.4.** Compute $\displaystyle\int_0^1 f(x) \, \mathrm{d}x$, where

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathcal{C} \cap E \\ x^3, & \text{otherwise,} \end{cases}$$

where $E$ is a Borel non-measurable subset of $[0,1]$.

*Answer.* The Lebesgue measure of $\mathcal{C} \cap E$ is zero (that is $E$ is Borel non-measurable, is irrelevant), so the integral is equal to

$$\int_0^1 x^3 \, \mathrm{d}x = \frac{1}{4}.$$

□

**Exercise 12.5.** Compute $\displaystyle\int_0^1 f(x) \, \mathrm{d}x$, where

$$f(x) = \begin{cases} x^3, & \text{if } x < \frac{1}{3} \text{ and } x \notin \mathbb{Q} \\ x^2, & \text{if } x \geq \frac{1}{3} \text{ and } x \notin \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{Q}. \end{cases}$$

*Answer.* Ignoring set of measure zero, we have

$$\int_0^{\frac{1}{3}} x^3 \,\mathrm{d}x + \int_{\frac{1}{3}}^1 x^2 \,\mathrm{d}x = \frac{35}{108}.$$

□

**Exercise 12.6.** Compute $\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x$, where

$$f(x) = \begin{cases} \sin x, & \text{if } \cos x \in \mathbb{Q} \\ \sin^2 x, & \text{if } \cos x \notin \mathbb{Q}. \end{cases}$$

*Answer.* Since cos is bijective on $[0, \frac{\pi}{2}]$, the set $\cos^{-1}(\mathbb{Q}) \cap [0, \frac{\pi}{2}]$ is countable and hence has measure zero, and the integral is equal to

$$\int_0^{\frac{\pi}{2}} \sin^2 x \,\mathrm{d}x = \left(\frac{x}{2} - \frac{\sin(2x)}{4}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

□

**Exercise 12.7.** Compute $\int f \,\mathrm{d}\delta_x$, where $f$ is a non-negative measurable function, and $\delta_x$ is the Dirac measure over point $x$.

*Answer.* Let $h$ be an elementary function. Then $\sum z\delta_x\{f = z\} = f(x)$. By definition, the integral in question is equal to $\sup f(x)$ over all elementary $h \le f$, and hence is equal to $f(x)$. □

**Exercise 12.8.** Compute $\int_0^{10} [x] \,\mathrm{d}x$.

*Answer.* By Theorem 9.2, the integral is equal to

$$0 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 + \cdots + 9 \cdot 1 = 45.$$

□

**Exercise 12.9.** Compute $\int_0^{2\pi} [\sin x] \,\mathrm{d}x$.

*Answer.* In the given range, the function $[\sin x]$ assumes the value 0 on $[0, \pi]$ (except of the point $\frac{\pi}{2}$ which may be ignored as a set of measure zero), and the value $-1$ on $(\pi, 2\pi)$. Hence by Theorem 9.2,

$$\int_0^{2\pi} (-[\sin x]) \,\mathrm{d}x = 1 \cdot (2\pi - \pi) = \pi,$$

and the integral in question is equal to $-\pi$. □

**Exercise 12.10.** Compute $\int_0^{\infty} \frac{1}{[x]!} \,\mathrm{d}x$.

*Answer.* By Theorem 9.2, the integral is equal to

$$\frac{1}{0!} \cdot 1 + \frac{1}{1!} \cdot 1 + \frac{1}{2!} \cdot 1 + \cdots = e.$$

□

**Exercise 12.11.** Compute $\lim_{n\to\infty} \int_1^{\infty} \frac{\sqrt{x}}{1 + nx^3} \,\mathrm{d}x$.

*Answer.* Using either Monotone Convergence Theorem, or Dominated Convergence Theorem, we may interchange lim and $\int$, and hence the limit in question is equal to

$$\int_1^{\infty} \left(\lim_{n\to\infty} \frac{\sqrt{x}}{1 + nx^3}\right) \mathrm{d}x = \int_1^{\infty} 0 \,\mathrm{d}x = 0.$$

□

Connection between Lebesgue and Riemann integrals (according to [BK, Chap. 5, Proposition 5.10] – without proof.

## Lecture 13. Discussion of homeworks

The following topic was included at 2016/2017, but not covered at 2018/2019: Hölder's inequality (according to [BK, Chap. 5, Proposition 5.6]).

## References


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