Category Theory

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Version of November 14, 2020

Literature

- J. Adamek, H. Herrlich, and G.E. Strecker, *Abstract and Concrete Categories. The Joy of Cats*, Online edition, 2004 (referred as Adamek et al.)
- G.M. Bergman, *An Invitation to General Algebra and Universal Constructions*, 2nd ed., Springer, 2015 (referred as Bergman)
- S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Springer, 1978. (referred as Mac Lane)
- S. Mac Lane and G. Birkhoff, *Algebra*, 3rd ed., AMS Chelsea, 1999 (referred as Mac Lane–Birkhoff)
- I.R. Shafarevich, Basic Notions of Algebra, Springer, 1990 (referred as Shafarevich)

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1.

Definition of category, motivation. Examples of categories

Motivation

From the previous courses you (suppose to) know that:

- A linear map is a map between two vector spaces which preserves linearity.
- A group homomorphism is a map between two groups which preserves the group multiplication, the neutral element, and the inverse operation.
- A (commutative) ring homomorphism is a map between two (commutative) rings which is additive and preserves the ring multiplication.
- A continuous map is a map between two topological spaces which preserves the topological structure (i.e., open sets).

Do you see the pattern?

Moreover, many statements about those maps (for example, composition of homomorphisms is a homomorphism) are formulated and proved exactly in the same way in all these cases. (For more such examples, see BERGMAN, pp. 213–217, MAC LANE, pp. 1–5, and SHAFAREVICH, pp. 202–204).

Definition

A *category* C consists of a class $Obj(\mathrm{C})$ whose elements are called *objects*, and a class $hom(\mathrm{C})$ whose elements are called *morphisms* (or *arrows*), such that there is a map

$$hom : Obj(\mathrm{C}) \times Obj(\mathrm{C}) \to \text{subsets of } hom(\mathrm{C})$$

satisfying the following axioms:

(i) (Existence of composition) For any $X, Y, Z \in Obj(\mathrm{C})$, there is a map $\circ$, called a *composition*

$$\circ : hom(Y, Z) \times hom(X, Y) \to hom(X, Z)$$

(ii) (Associativity) If $X, Y, Z, W \in Obj(\mathrm{C})$, and $h \in hom(Z, W)$, $g \in hom(Y, Z)$, $f \in hom(X, Y)$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(iii) (Existence of identity) For any $Y \in Obj(\mathrm{C})$, there exists a morphism $1_Y \in \mathrm{hom}(Y, Y)$ such that $1_Y \circ f = f$ for any $f \in hom(X, Y)$, and $g \circ 1_Y = g$ for any $g \in hom(Y, Z)$.

Examples of categories

Examples

- All sets (objects) and maps between them (morphisms) form the category of sets Set.
- The classes mentioned at the first slide form respectively: the category of vector spaces Vect, the category of groups Group, the category of rings Ring, the category of commutative rings CommRing, and the category of topological spaces Top.

For more examples, see ADAMEK AT AL., pp. 22–24,
BERGMAN, pp. 221–226, MAC LANE, pp. 10–12,
MAC LANE–BIRKHOFF, pp. 496,498,
SHAFAREVICH, pp. 205–206. See also an impressive list of all categories mentioned in ADAMEK ET AL., pp. 475–479.

Exercise

Why in the definition of category we are speaking about a “class” of objects and not about a set of objects?

Dual category

Definition

For a category C, the *dual* (or *opposite*) *category* C^{op} is the category having the same objects as C, and for which $hom_{\mathsf{C}^{\mathsf{op}}}(X, Y) = hom_{\mathsf{C}}(Y, X)$, and $f \circ^{op} g = g \circ f$.

Informally, the dual category has the same morphisms, but in the "opposite" directions.

Example

If C is the category of ordered sets with the relation $\leq$, then C^{op} is the category of ordered sets with the relation $\geq$.

Subcategories

Informally, a *subcategory* of a category C a subclass of objects of C, "closed" with respect to composition of morphisms.

Exercise

Give the precise definition of a subcategory of a category.

Hint: see MAC LANE–BIRKHOFF, p. 498.

Examples

- Category of abelian groups AbGroup is a subcategory in Group.
- **CommRing**, and the category of fields are subcategories in **Ring**.

Product of categories

The notion of cartesian product of two sets is readily extended to the case of categories.

Definition

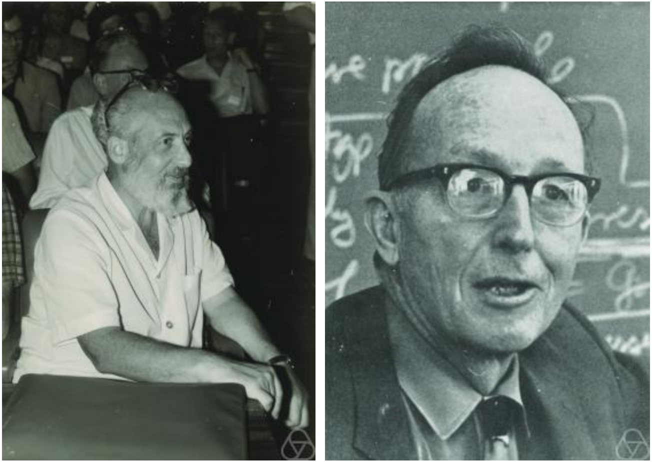
A *product* of two categories B and C, denoted by B $\times$ C, is defined as a category whose objects are $Obj(\mathrm{B}) \times Obj(\mathrm{C})$, and whose arrows are $hom(\mathrm{B}) \times hom(\mathrm{C})$, and composition of arrows is performed component-wise:

$$(f, g) \circ (f', g') = (f \circ f', g \circ g')$$

for suitable $f, f' \in hom(\mathrm{B})$ and $g, g' \in hom(\mathrm{C})$.

A bit of history

Category theory was created by Samuel Eilenberg (1913–1998) and Saunders Mac Lane (1909–2005) around 1942–1945.



“The devious and sophisticated European versus the innocent but honest American?” (D. Eisenbud, from the preface to “A Mathematical Autobiography” by Saunders Mac Lane).

2.

Functors

Covariant functor

A functor is, an essence, a morphism (i.e., a map “preserving the structure”) of categories.

Definition

A *covariant functor* (or just *functor*) from a category C to a category D consists of two maps (denoted by abuse of notation by the same letter), $F : Obj(\mathrm{C}) \to Obj(\mathrm{D})$ and $F : hom(X, Y) \to hom(F(X), F(Y))$ for any $X, Y \in Obj(\mathrm{C})$ satisfying the axioms:

(i) $F(1_X) = 1_{F(X)}$ for any $X \in Obj(\mathrm{C})$.

(ii) $F(f \circ g) = F(f) \circ F(g)$ for any morphisms f, g whenever $f \circ g$ is defined.

Contravariant functor, bifunctor

Definition

A *contravariant functor* from a category C to a category D is obtained from the previous definition by replacing the second F by $F : hom(X, Y) \to hom(F(Y), F(X))$, and the second axiom by:

(ii) $F(f \circ g) = F(g) \circ F(f)$.

Exercise

Rewrite the definitions of covariant and contravariant functor in terms of commutative diagrams.

Hint: see Mac Lane–Birkhoff, pp. 131–132,504–505.

Definition

A *bifunctor* from a pair of categories B, C to a category D is a functor from B $\times$ C to D.

Examples

- The map assigning to a vector space its n-fold tensor product, is a covariant functor from **Vect** to itself. (Prove this!)
- The map assigning to a commutative ring A a group (say) $SL_n(A)$ is a functor from CommRing to **Group**.
- *Forgetful functors*, where a part of the structure of the objects is "forgotten", for example, the functor from **Group** to **Set**, sending a group to the underlying set.
- The cartesian product of two sets is a bifunctor Set $\times$ Set $\to$ Set.
- The map sending a vector space to its adjoint is a contravariant functor.

For more examples, see ADAMEK AT AL., pp. 30–32, BERGMAN, pp. 239–241, MAC LANE, pp. 13–14,35, MAC LANE–BIRKHOFF, pp. 131–133,501–503,505–506, SHAFAREVICH, pp. 208–213.

3.

Equivalence of categories

Isomorphism of categories

Definition

Two categories C and D are said *isomorphic*, if there exists a functor $F : \mathsf{C} \to \mathsf{D}$, called *isomorphism*, such that there is an "inverse" functor $F^{-1} : \mathsf{D} \to \mathsf{C}$: $F^{-1} \circ F = \mathrm{id}_{\mathsf{C}}$ and $F \circ F^{-1} = \mathrm{id}_{\mathsf{D}}$.

Example

- The category of Boolean algebras is isomorphic to the category of Boolean rings.
- The category of left R-modules over a commutative ring R is isomorphic to the category of right R-modules.

For more examples of isomorphic categories, see ADAMEK ET AL., pp. 33–34.

Faithful and full functors

Isomorphic categories are, essentially, “the same”, and the concept of isomorphism of categories is very restrictive. The less weaker concept of equivalence of categories turns out to be more meaningful.

Let C, D be two categories, and $F : \mathsf{C} \to \mathsf{D}$ a functor between them. For any two objects $X, Y \in Obj(\mathsf{C})$, consider the hom-set restriction

$$F : hom_{\mathsf{C}}(X, Y) \to hom_{\mathsf{D}}(F(X), F(Y)).$$

Definition

1. A functor is called *embedding* if it is injective on morphisms.
2. A functor is called *faithful* if all its hom-set restriction are injective.
3. A functor is called *full* if all its hom-set restrictions are surjective.

Examples

The forgetful functor Vect $\to$ Set is faithful, but is neither full nor an embedding.

For further examples of embeddings, faithful, and full functors, see ADAMEK ET AL., pp. 34–35, and BERGMAN, p. 244.

Equivalence of categories

Definition

Two categories C and D are called *equivalent*, if there is a functor $F : \mathsf{C} \to \mathsf{D}$ which is faithful and full, and for any object $Z \in Obj(\mathsf{D})$, there is an object $X \in Obj(\mathsf{C})$, such that $F(X) \simeq Z$.

Examples

- The category of matrices is equivalent of the category of Vect, but not isomorphic to it.
- The category of finite-dimensional real vector space is equivalent to its dual (each vector space is mapped to its adjoint).

For details and more examples of isomorphic and equivalent categories, see ADAMEK ET AL., pp. 36,38.

4.

Small and large categories, concrete categories

Small categories

Definition

A category is called *small* if the class of its objects is a set, and *large* otherwise.

Lemma

The class of morphisms in a small category is a set too.

Examples

- The category of matrices is small.
- The category of (all) monoids is large.

Exercise

Which of the categories considered so far are small?

For more examples of small and large categories, see ADAMEK ET AL., p. 39 and MAC LANE, pp. 24–26.

The category **Cat**

Lemma

All small categories form a category Cat. The morphisms are functors between categories.

Exercise 1

Is Cat small?

Exercise 2

Can we speak about category of all (not necessarily small) categories?

Hint: see ADAMEK ET AL., p. 39.

Concrete categories

If we consider categories comprised of “concrete” objects, like vector or topological spaces, we loose some information, as the emphasis in “abstract” categories is not on objects themselves, but on relationship between them. The notion of concrete category aims to rectify this deficiency.

Definition

A category C is called *concrete* if there is a faithful functor (called the *forgetful functor*) $\mathsf{C} \to \mathsf{Set}$.

Exercise

Which of the categories considered so far are concrete?

Theorem

Every small category can be turned into a concrete one, i.e. admits a faithful functor to the category of small sets.

For more examples of concrete categories, see ADAMEK ET AL., p. 62, and MAC LANE–BIRKHOFF, pp. 142–143,497.

Concrete functors

Definition

Let C and D be two concrete categories, with the corresponding forgetful functors $U : \mathsf{C} \to \mathsf{Set}$ and $V : \mathsf{D} \to \mathsf{Set}$. A functor $F : \mathsf{C} \to \mathsf{D}$ is called *concrete*, if $U = V \circ F$.

Lemma

Every concrete functor is faithful.

Example

The forgetful functor from the concrete category of rings to the concrete category of abelian groups which "forgets multiplication", is concrete.

For more examples, see ADAMEK ET AL., p. 66.

5.

Natural transformations

Natural transformations

The same way as functor provides a "morphism" between categories, natural transformation provides "morphism" between functors.

Definition

A *natural transformation* between two functors F, G from a category C to a category D, is a map $\tau : Obj(\mathrm{C}) \to hom(\mathrm{D})$, $X \mapsto \tau_X$, such that for any $X, Y \in Obj(\mathrm{C})$, and any arrow $f \in hom(X, Y)$, the following diagram

$$\begin{tikzcd}
F(X) \arrow[r, "\tau_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\tau_Y"] & G(Y)
\end{tikzcd}$$

is commutative.

If each τ_X is invertible, then τ is called a *natural equivalence*.

The set of all natural transformations between F and G is denoted by $[F, G]$.

Examples of natural transformations

1. The determinant, considered as a map $det : GL_n \to ()^*$, is a natural transformation between two functors from CommRing to Group.
2. Abelianization of a group, i.e. the natural projection $G \to G/[G, G]$ for a group G, is a natural transformation between two functors from Group to Group.

For details and further examples, see ADAMEK ET AL., pp. 83–85, BERGMAN, p. 280, MAC LANE, pp. 16–18, and MAC LANE–BIRKHOFF, pp. 507–508.

6.

Universal constructions, limits, colimits

Universal arrow

Definition

If $S : \mathsf{D} \to \mathsf{C}$ is a functor between two categories D and C, and $c \in Obj(\mathsf{C})$, a *universal arrow* from c to S is a pair (r, u) consisting of an object $r \in Obj(\mathsf{D})$, and an arrow $u : c \to S(r)$ of C, such that to every pair (d, f) with $d \in Obj(\mathsf{D})$ and $f : c \to S(d)$ an arrow of C, there is a unique arrow $f' : r \to d$ of D with $S(f') \circ u = f$.

Examples

- A map sending an element of a base of a vector space, considered as a set, to the same vector space, considered as an element of Vect.
- A map sending an integral domain to its field of quotients.

For details and other examples, see Bergman, pp. 295–296, Mac Lane, pp. 56–57, and Mac Lane–Birkhoff, pp. 130–131.

Universal element

An important particular case of an universal arrow is universal element.

Definition

If D is a category and $H : \mathrm{D} \to \mathrm{Set}$ a functor, a *universal element* of the functor H is a pair (r, e) consisting of an object $r \in \mathrm{D}$ and an element $e \in H(r)$ such that for every pair (d, x) with $d \in Obj(\mathrm{D})$ and $x \in H(d)$, there is a unique arrow $f : r \to d$ of D with $(H(f))(e) = x$.

Examples

Partition of a set into equivalence classes, quotients of a group by a normal subgroup, and tensor products can be expressed in terms of a universal element in appropriate categories. For details and other examples, see MAC LANE, pp. 57–58.

Limit

Important instances of universal constructions are limits and colimits.

Definition

Let $F : \mathrm{D} \to \mathrm{C}$ be a functor between two categories D, C. A *limit* of F, denoted by $\varprojlim F$, is an object $L \in Obj(\mathrm{C})$ such that for every $X \in Obj(\mathrm{D})$ there is a morphism $p(X) : L \to F(X)$ satisfying the following property: for $f \in hom_{\mathrm{D}}(X, Y)$, one has $p(Y) = F(f)p(X)$. Moreover, p is universal for this property, i.e., given any object $M \in Obj(\mathrm{C})$, and family of morphisms $m(X) : M \to F(X)$, which similarly make commuting triangles with the morphisms $F(f)$, there exists a unique morphism $h : M \to L$ such that for all X, $m(X) = p(X) \circ h$.

Examples of constructions described in terms of limits

- p-adic numbers, see BERGMAN, pp. 317–323 or MAC LANE, pp. 110-111.
- Formal power series.

Colimit

Reversing arrows, we get the dual notion:

Definition

Let $F : \mathrm{D} \to \mathrm{C}$ be a functor between two categories D, C. A *colimit* of F, denoted by $\varinjlim F$, is an object $L \in Obj(\mathrm{C})$ such that for every $X \in Obj(\mathrm{D})$ there is a morphism $q(X) : F(X) \to L$ satisfying the following property: for $f \in hom_{\mathrm{D}}(X, Y)$, one has $q(X) = q(Y)F(f)$. Moreover, q is universal for this property, i.e., given any object $M \in Obj(\mathrm{C})$, and family of morphisms $m(X) : F(X) \to M$, which similarly make commuting triangles with the morphisms $F(f)$, there exists a unique morphism $h : L \to M$ such that for all X, $m(X) = h \circ q(X)$.

Warning: limits and colimits not always exist!

One of the main questions related to limits and colimits is when that or another functor preserves them. See ADAMEK ET AL., pp. 223–226, BERGMAN ET AL., pp. 347–348,352 or MAC LANE, pp. 116–118 for details.

Direct and inverse limits

Important particular cases of limit and colimit are *inverse* and *direct* limit, respectively.

inverse limit: $\dots \leftarrow C_{n-1} \leftarrow C_n \leftarrow C_{n+1} \leftarrow \dots$

direct limit: $\dots \rightarrow C_{n-1} \rightarrow C_n \rightarrow C_{n+1} \rightarrow \dots$

An example of inverse limit: direct product $\prod_{i \in I} A_i$.
An example of direct limit: direct sum $\bigoplus_{i \in I} A_i$.

A well known theological concept is that of the transcendental divine consciousness as a limit of restricted human consciousnesses. In this setup, optimist would say that this limit is a direct limit, while pessimist would say that this is an inverse one. (As seen somewhere on mathoverflow).

Exercise

Rewrite all definitions from this section in terms of commutative diagrams.

Hint: See Mac Lane, p. 55.

7.

The Yoneda Lemma

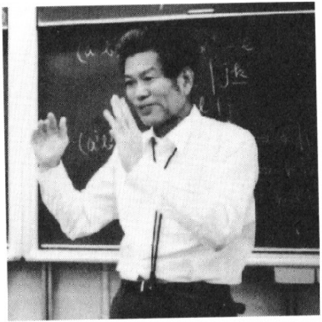
The Yoneda Lemma

Lemma

Let C be a category, $F : \mathsf{C} \to \mathsf{Set}$ a functor, and $X \in Obj(\mathsf{C})$. Then the map

$$\begin{aligned} [hom(X,-),F] &\to F(X) \\ \sigma &\mapsto \sigma_X(\mathrm{id}_X) \end{aligned}$$

is a bijection



米田信夫

Nobuo Yoneda (1930–1996)

The Yoneda Lemma (cont.)

Roughly, the Yoneda Lemma says that an object in a category is determined by the functor that records morphisms from each of the objects of the category (or, the object is best understood in the context of a category surrounding it).

The proof if the Yoneda Lemma uses the concept of universal arrows, see MAC LANE, pp. 59–61 for details.

Exercise

The Yoneda Lemma is formulated for covariant functor F. Formulate and prove the version of the Lemma for contravariant functor.

Hint: see BERGMAN, pp. 300–301.

The category of functors

Notation

For two categories C, D, denote by $[C, D]$ the set of all functors from C to D.

Theorem

If C and D are small, then $[C, D]$ forms a category, with functors being natural transformations between functors.

Exercise

Does $[C, D]$ forms a category in the similar way for arbitrary, not necessarily small C and D?

An embedding theorem

Theorem

For any category C, the functor $E : \mathsf{C} \to [\mathsf{C}^{op}, \mathsf{Set}]$, defined by

$$E(X \xrightarrow{f} Y) = hom(-, X) \xrightarrow{\sigma_f} hom(-, Y),$$

where $\sigma_f(g) = f \circ g$, is a full embedding.

Proof: This is an (easy) corollary of the Yoneda Lemma.

This theorem is a vast generalization of the theorem from group theory about embedding of any group in a symmetric group, and similar results.

8.

Adjoint functors

Adjoint functors

Definition

Let C, D be two categories, and $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{C}$ two functors in opposite directions between them. The functor F is called a left adjoint to G, and G is called a right adjoint to F, if for any objects $X \in Obj(\mathsf{C})$ and $Y \in Obj(\mathsf{D})$, there is a bijection of sets

$$hom(F(X), Y) \simeq hom(X, G(Y)),$$

natural in the arguments X and Y.

Exercise

Rewrite this definition using the notions of universal arrows or universal elements.

Hint: See Adamek et al., p. 305, or Bergman, p. 309, or Mac Lane, pp. 81–82.

Warning

Left/right adjoint functors not always exist!

Examples of adjoint functors

Examples

- Tensor product and Hom in the category of modules over a (commutative) ring.
- The functor Set → Top supplying each set with the discrete topology, and the forgetful functor Top → Set.
- The forgetful functor Group → Set, and the functor Set → Group assigning to a set X the free group freely generated by X.

For details and other examples, see ADAMEK ET AL., pp. 305,319, BERGMAN, pp. 311–312, MAC LANE, pp. 87,123–125, and MAC LANE–BIRKHOFF, p. 519.

Properties of adjoint functors

Theorem 1

Any two left(right)-adjoints of a given functor are naturally isomorphic.

Theorem 2

The composition of adjoint functors is adjoint.

Theorem 3

Adjoint functors preserve limits.

9.

Monoidal categories

Definition of a monoidal category

A *monoidal category* is a category C equipped with bifunctor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, and an object $I \in \mathsf{C}$, called the *unit* (or *identity*) *object* satisfying the following conditions:

1. (Associativity) There is a natural (in three arguments A, B, C) isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes \otimes C \simeq A \otimes (B \otimes C)$.
2. (Identity) There are two natural isomorphisms $\lambda_A : E \otimes A \simeq A$ and $\rho_A : A \otimes E \simeq A$.
3. (Coherence) For any $A, B, C, D \in Obj(\mathsf{C})$, the pentagonal diagram

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes 1_D"] \arrow[d, "\alpha_{A \otimes B, C, D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A, B \otimes C, D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

commutes.

to be continued ...

Definition of a monoidal category (cont.)

5. (Coherence) For any $A, B, C \in Obj(\mathsf{C})$, the triangle diagram

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

commutes.

Examples of monoidal categories

Informally, a category is monoidal if it is equipped with a "product" which is associative up to isomorphism.

Examples

1. Set with respect to cartesian product.
2. Vect, and, more generally, the category of modules over a fixed commutative ring, with respect to tensor product.
3. The category of associative algebras with respect to tensor product.
4. Top with respect to the product of topological spaces.
5. Cat with respect to the product of categories.

Exercise

What will serve as a unit in each of these examples?

10.

Categorification

Categorification

Categorification is a process of replacing set-theoretic concepts and statements by their category-theoretic analogues. It allows to reveal hidden structures in mathematics, and bring them to a newer level of understanding.

set-theoretic notion	**category-theoretic counterpart**
set	category
elements	objects
function	functor
equation	natural transformation

Examples

- Natural numbers $\rightsquigarrow$ Cardinalities of finite sets.
- Symmetric functions $\rightsquigarrow$ Representations of the symmetric group.
- Monoid (a set with an associative binary operation and a unit) $\rightsquigarrow$ Monoidal category.

Another (historical) example

One of the earlier examples of categorification is the replacement of Betti numbers by (co)homology groups (whose ranks are Betti numbers), done by Emmy Noether in 1920s-1930s. This gave birth to the homological algebra.

$$b_i = \operatorname{rk} H_i(X, \mathbb{Q})$$

$b_0, b_1, b_2, \cdots \rightsquigarrow$

$\rightsquigarrow H_0, H_1, H_2, \ldots$

11.

Applications in computer science (functional programming, database design)

Applications in functional programming

Category theory, due its generality and flexibility, is vastly applicable in computer science. Below are just a few examples.

A central concept in Haskell and other functional programming languages, used in sequential computations, is that of monad which comes from category theory. Roughly, a *monad* is a categorical generalization of a closure operator on a partially ordered set. Monad is a functor from a category to itself, equipped with two natural transformations, which give it a monoid-like structure. For an exact definition, see MAC LANE, pp. 137–138.

Applications in database design

1. Databases, and, more generally, knowledge bases, can be represented as a special kind of automata: a database query brings the automaton to another state, producing the answer to the query. One of important and complicated question in the theory of databases is whether two databases are, essentially, the "same", i.e. produce the same answers to the same queries. This question may be approached using the representation above, considering the category of all databases as a subcategory of the category of automata, and employing the notion of equivalence of categories.
2. Alternatively, database schemas may be represented as categories, with functors representing migration from one schema to another (a task frequently needed to be performed on practice).
3. For finite state machines, "minimal realization" and "behavior" could be considered as adjoint functors. See MAC LANE, p. 89 for details.

12.

2-categories and applications in physics (string theory, topological quantum field theory)

Braided categories in physics

A *braided category* is a monoidal category equipped with braiding, i.e. the commutativity natural isomorphism $\gamma_{A,B} : A \otimes B \to B \otimes A$ satisfying additional identities which are satisfied in the braid group.

In string theory, particles are represented as strings weaving around each other, so the concepts of braids and of braided category are applicable. See Mac Lane, pp. 260–266 for details.

2-categories

An ordinary category has objects and morphisms (1-morphisms). A 2-category extends this by including “morphisms between morphisms” (2-morphisms). Thus, in a sense, 2-categories are categorifications of ordinary categories. See Mac Lane, pp. 272–279 for details.

Example

Cat is actually a 2-category.

2-categories is another categorical concept used in string theory. Transformations of strings, which can be considered as morphisms in an appropriate category, as they move along surfaces in spacetime, can be considered as 2-morphisms:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D}] & \bullet
\arrow[Rightarrow, from=U, to=D]
\end{tikzcd}$$

The End