$$\underbrace{\mathbb{R} \subset \mathbb{C}}_{\text{associative, commutative}} \subset \mathbb{H} \subset \ldots$$

1, 2, 4, 8

$\mathbb{H}$: associative $(xy)z \overset{?}{=} x(yz)$, not commutative

$\langle 1 \rangle$ $\quad$ $\langle 1, i \rangle$ $\quad$ $\langle 1, i, j, k \rangle$

$\mathbb{Q}$

$a \cdot 1 + bi$, $a, b \in \mathbb{R}$, $i^2 = -1$

$i^2 = j^2 = k^2 = -1$

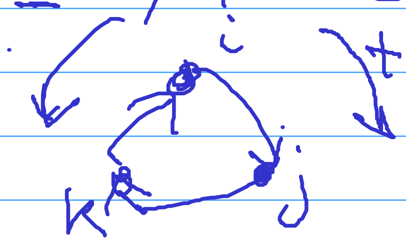
$\langle 1, i \rangle \cong \mathbb{C}$

$\langle 1, j \rangle \cong \mathbb{C}$

$\langle 1, k \rangle \cong \mathbb{C}$

$$\boxed{x^{-1} \cdot x = x \cdot x^{-1} = 1, \quad x \neq 0}$$

$1 \cdot x = x \cdot 1$



$ij = k$

$jk = i$

$ki = j$

$ik = -j$

$kj = -i$

$ji = -k$

$$\mathbb{H}_{\mathbb{C}} = \langle 1, i, j, k \rangle$$

Theorem. $\mathbb{H}_{\mathbb{C}} \cong M_2(\mathbb{C})$.

(isomorphism of algebras)

Proof. $f : \mathbb{H}_{\mathbb{C}} \to M_2(\mathbb{C})$

$\langle 1, i, j, k \rangle_{\mathbb{C}}$

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

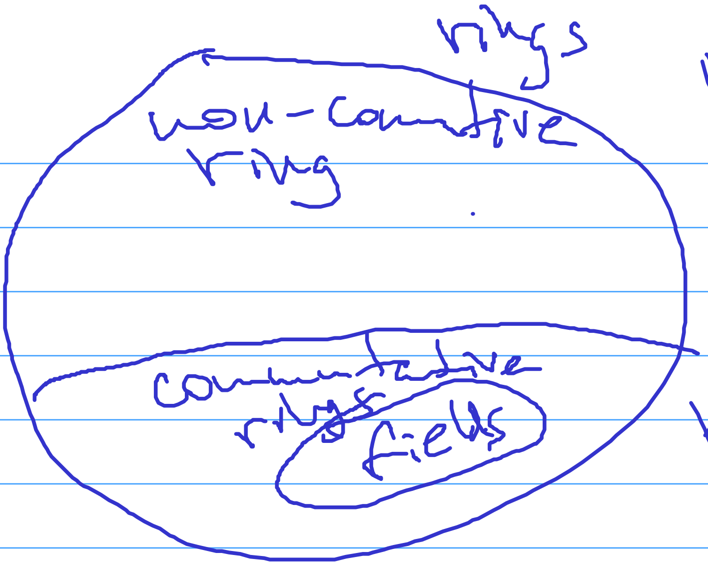
$f(ij) \overset{?}{=} f(i) \cdot f(j)$

$f(i^2) \overset{?}{=} f(i)^2$

$$f(k) \overset{?}{=} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$a, b, c, d \in \mathbb{C}$

$$a1 + bi + cj + dk \mapsto a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$



$\boxed{\langle F, +, \cdot, 0, 1 \rangle}$

$a \cdot 1 = \cancel{1a} = a$

$\forall a \in F$, $a \neq 0$, $\exists a^{-1} \in F$: $a \cdot a^{-1} = 1$

$\cancel{0 \cdot x = 0}$

$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ $\boxed{\text{fields}}$

$\{0, 1\}$

$GF(2)$ $\mathbb{R}^*$

$\langle F, +, \cdot, 0, 1 \rangle$

$\langle F \setminus \{0\}, \cdot, 1 \rangle = F^*$ abelian group

$\boxed{GF(2)} = \{0, 1\}$

$2 = 0$

$\boxed{1 + 1 = 0}$

$1^{-1} = 1$

$S \subset F$ subfield

$S + S = S$ $x = x + 0$

$S \cdot S = S$ $x = x \cdot 1$

$0, 1 \in S$

Isomorphism of fields

$f: F \to K$

$f(0) = 0$ $\ker f =$

$f(1) = 1$

$f(a+b) = f(a) + f(b)$

$f(ab) = f(a) \cdot f(b)$

$f(a^{-1}) = f(a)^{-1}$

$f: F \to K$ $\quad$ $\ker f \trianglelefteq F$

$f(\cancel{x}) = 0 \quad x \in F$ $\quad$ $\ker f = 0$

$f(x) \cdot f(y) = f(xy) = 0 \Rightarrow xy \in \ker f$

$\forall y \in F$

$f(1) = 1 \neq 0$

$\mathbb{Q}$ $\mathbb{C}$

Proof of theorem in p. 34 at the dida

$\Leftarrow$ F - field $\quad I \trianglelefteq F$

$I \neq 0 \quad \cancel{x} \in F, x \neq 0 \quad I \cdot F \subseteq F \quad \forall x \in F$

$\Downarrow \quad x \cdot x^{-1} = 1 \quad 1 \cdot x = x$

$I = F$

$\Rightarrow$ F simple commutative ring with 1

principal ideal

$\forall x \in F$, $x \neq 0$: $xF \trianglelefteq F$ $\quad$ $xF = \{\cancel{0}\}$ $\quad \ni 1$

$(xa) \cdot b = x(ab)$ $\quad$ $x \cdot 1 = x \in xF \neq 0$

i.e. F

$\exists y: 1 = xy \quad y = x^{-1}$

$\mathbb{R}[x]$ $\quad$ $f(x) g(x) = 1$

$(1 + x + x^2)(\quad) = 1$