$\langle R, +, \cdot \rangle$ **Def.** $\operatorname{Ker}\varphi = \{x \in R \mid \varphi(x) = 0\}$

Th. 1) For any homomorphism of rings $\varphi: R \to S$

$\operatorname{Ker}\varphi \triangleleft R$

Proof. $\boxed{\varphi(xy) = \varphi(x)\varphi(y)}$

$\varphi(x+y) = \varphi(x) + \varphi(y)$

$\boxed{\varphi(0) = 0}$

$\varphi(-x) = -\varphi(x)$

$0 \in \operatorname{Ker}\varphi$

$x, y \in \operatorname{Ker}\varphi \Rightarrow x+y \in \operatorname{Ker}\varphi$

$x \in \operatorname{Ker}\varphi \Rightarrow -x \in \operatorname{Ker}\varphi$

$x \in \operatorname{Ker}\varphi, y \in R \Rightarrow xy \in \operatorname{Ker}\varphi$

$x \in R, y \in \operatorname{Ker}\varphi \Rightarrow$

2) $I \triangleleft R \Rightarrow \exists \varphi: R \to S : I = \operatorname{Ker}\varphi$

Proof. $\varphi: R \to R/I$

$x \mapsto x + I$

$I + I = I$

$0 + x = x$

$\varphi(x+y) = \varphi(x) + \varphi(y)$

$x + y + I = (x+I) + (y+I) =$

$= x + y + (I + I)$

$-I = I$

$\varphi(0) = 0 + I = I$

$I = -I$

$\boxed{x = -(-x)}$

$\varphi(-x) = -\varphi(x)$

$-x + I = -(x+I) = -x - I$

$$\varphi(xy) = \varphi(x)\varphi(y)$$

$$xy + I = (x+I)\cdot(y+I)$$

$$\varphi: R \to R/I$$

$$I = \{0\}$$

$$\operatorname{Ker}\varphi = I$$

$$0 \lhd R$$

$$R \lhd R$$

$$x \in R \quad \varphi(x) = 0 \iff x \in I$$

$$\varphi(I) = 0$$

$$x + I = I \Rightarrow x \in I$$

Homomorphism theorems for rings

Th. If $\varphi: R \to S$ is a homomorphism of rings, then

$$R/\operatorname{Ker}\varphi \cong \operatorname{Im}\varphi.$$

Proof 1) $\operatorname{Im}\varphi$ is a subring.

$$a, b \in \operatorname{Im}\varphi \Rightarrow ab \in \operatorname{Im}\varphi.$$

$$\underbrace{\varphi(xy)}_{\in \operatorname{Im}\varphi} = \underbrace{\varphi(x)}_{a}\underbrace{\varphi(y)}_{b} \quad \forall x, y \in R$$

2) $f: R/\operatorname{Ker}\varphi \to \operatorname{Im}\varphi$

$$x + \operatorname{Ker}\varphi \mapsto \varphi(x) \qquad x, y \in R$$

$$f((x+\operatorname{Ker}\varphi)\cdot(y+\operatorname{Ker}\varphi)) \stackrel{?}{=} f(x+\operatorname{Ker}\varphi)\cdot f(y+\operatorname{Ker}\varphi)$$

$$f((x+\operatorname{Ker}\varphi)\cdot(y+\operatorname{Ker}\varphi)) = f(x+\operatorname{Ker}\varphi)\cdot f(y+\operatorname{Ker}\varphi)$$

$$f((x+\operatorname{Ker}\varphi)\cdot(y+\operatorname{Ker}\varphi)) = f(xy+\operatorname{Ker}\varphi) = \varphi(xy)$$

$$f(x+\operatorname{Ker}\varphi)\cdot f(y+\operatorname{Ker}\varphi) = \varphi(x)\cdot\varphi(y)$$

$$f: R/\operatorname{Ker}\varphi \longrightarrow \operatorname{Im}\varphi$$

$$x+\operatorname{Ker}\varphi \longmapsto \varphi(x)$$

$$x'+\operatorname{Ker}\varphi = x+\operatorname{Ker}\varphi, \quad \varphi(x') = \varphi(x)$$

$$x'-x \in \operatorname{Ker}\varphi$$

$$\varphi(x'-x)=0 = \varphi(x)-\varphi(x')$$

$x, y \in R$

$$f(x+\operatorname{Ker}\varphi) = f(y+\operatorname{Ker}\varphi)$$

$$\varphi(x) = \varphi(y)$$

$$\varphi(x-y)=0$$

$$x-y \in \operatorname{Ker}\varphi$$

$$x+\operatorname{Ker}\varphi = y+\operatorname{Ker}\varphi$$

$$\operatorname{Im} f \overset{\supseteq}{\subseteq} \operatorname{Im}\varphi$$

$$\forall a \in \operatorname{Im}\varphi \ \exists\, a+\operatorname{Ker}\varphi \in R/\operatorname{Ker}\varphi :$$

$$f(a+\operatorname{Ker}\varphi) = a$$

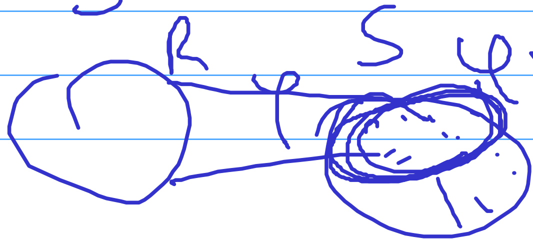
Reformulation of the 1st hom. theorem for rings. $\square$

For any surjective hom. $\varphi: R \to S$

$$R/\operatorname{Ker}\varphi \cong S$$

$\cancel{S}$ $\operatorname{Im}\varphi$

$\varphi: R \to \operatorname{Im}\varphi$



$p: \mathbb{Z} \to GF(2) = \{\bar{0}, \bar{1}\}$

$\operatorname{Ker} p = 2\mathbb{Z} \triangleleft \mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z} \cong GF(2)$

$ev_x : Map(\mathbb{R}) \to \mathbb{R}$ $\quad x \in \mathbb{R}$

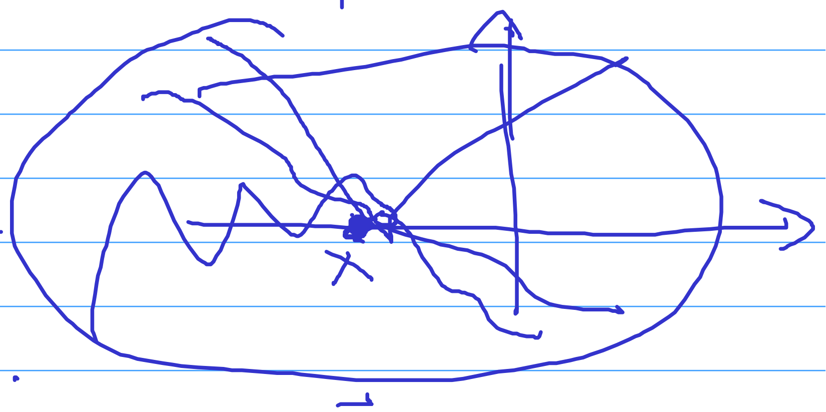
$f \mapsto f(x)$ $\quad ev_x(f) = 0$

$\operatorname{Ker}(ev_x) = \{ f \in Map(\mathbb{R}) \mid f(x) = 0 \}$

$\triangleleft\, Map(\mathbb{R})$ $\quad f(x) \cdot f(y) = 0$

$Map(\mathbb{R}) / \operatorname{Ker}(ev_x)$

$\mathbb{R} \setminus \{x\} \to \mathbb{R}$



$\{\bar{0}, \bar{1}\}$ $\quad R^2 = RR = 0$.

$I \triangleleft J \triangleleft R \not\Rightarrow I \triangleleft R$

$I \triangleleft R$

$n\mathbb{Z}/m\mathbb{Z}$

$I \triangleleft R, \ S \subseteq R$

R/I

S/I

$S/S \cap I \cong S+I/I$