$$I \triangleleft J \triangleleft R \not\Rightarrow I \triangleleft R$$

$$IJ \subseteq I \qquad JI \subseteq I$$

$$\boxed{\begin{array}{c} JR \subseteq J \\ RJ \subseteq J \end{array}} \overset{?}{\Rightarrow} \begin{array}{c} IR \subseteq I \\ RI \subseteq I \end{array}$$

$J I R$

~~$K = \mathbb{R}, \mathbb{C}, \mathbb{Q}, GF(2)$~~

$$R = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & h \end{pmatrix} \middle| \, a, b, \dots, h \in \mathbb{R} \right\}$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \cdot \end{pmatrix} \cdot \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \cdot \end{pmatrix}$$

$$J = \left\{ \begin{pmatrix} 0 & 0 & \cdot \\ 0 & 0 & \cdot \\ 0 & 0 & 0 \end{pmatrix} \right\} \qquad I = \left\{ \begin{pmatrix} 0 & 0 & \cdot \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$I \trianglelefteq J \triangleleft R \qquad J \subseteq R \qquad I \subseteq J$$

$$\begin{pmatrix} 0 & 0 & \cdot \\ 0 & 0 & \cdot \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \cdot \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdot \\ 0 & 0 & \cdot \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \cdot \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \cdot \\ 0 & 0 & \cdot \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I \ntriangleleft R$$

$$\begin{pmatrix} 0 & 0 & a^{\neq 0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ b_{\neq 0} & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} ab^{\neq 0} & & \cdot \\ & & \cdot \\ & & \end{pmatrix}$$

$\boxed{R = \mathbb{Q}[x]/(x^2)} = \langle 1, \bar{x} \mid \begin{smallmatrix} 1\cdot\bar{x}=\bar{x} \\ \bar{x}\cdot\bar{x}=0 \end{smallmatrix} \rangle =$

$= \{a + b\bar{x} \mid a, b \in \mathbb{Q}\}$

$a \in R \quad \boxed{(x^2) \ni x^2 f(x)}, \quad \cong \{(a, b) \mid a, b \in \mathbb{Q}\}$

(a) — ideal generated

$(a,b)+(a',b') = (a+a', b+b')$

$(a,b)\cdot(a',b') = (aa', ab'+a'b)$

$na, \; n \in \mathbb{Z}$

$a^2, a^3, \ldots, a^n$

$I = RaR = \langle \cancel{xay} \mid x, y \in R \rangle \quad a \cdot 1 \cdot R$

$Ra(RR) \subseteq RaR$

$J = \langle \bar{x} \rangle = \mathbb{Q}\bar{x} \qquad JR \subseteq J \qquad J \trianglelefteq R$

$(a + b\bar{x}) \cdot c\bar{x} = ac\bar{x} \in \mathbb{Q}\bar{x} = J$

$I = \mathbb{Z}\bar{x} = \{n\bar{x} \mid n \in \mathbb{Z}\}$

$\boxed{IJ = 0} \subseteq J \qquad I \trianglelefteq J \qquad \boxed{J^2 = 0}$

$n\bar{x} \cdot \mathbb{Q}\bar{x} = 0 \qquad I \ntrianglelefteq R$

$IR \overset{?}{\subseteq} I$

$n\bar{x} \cdot (a + b\bar{x}) = na\bar{x} \notin I$

$n \in \mathbb{Z}, \; a, b \in \mathbb{Q} \qquad \underset{\mathbb{Z}}{\mathbb{Q}}$

$\mathbb{Z} \ntrianglelefteq \mathbb{Q}$

Def. A ring R is simple if 0 and R are only ideals of R.

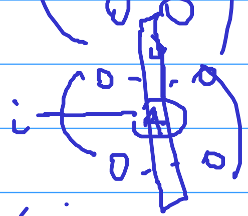
Theorem $M_n(R)$ is simple.

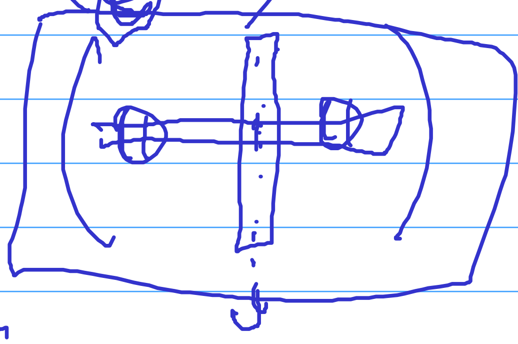
Sketch of the Proof. $I \trianglelefteq M_n(R)$ $\quad I \dashv I \subseteq I$

$n=2$ $\quad$ $I \cdot M_n(R) \subseteq I$ $\quad$ $0 \subsetneq I$

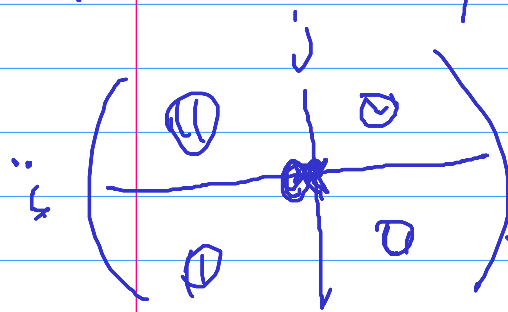
$M_n(R) \cdot I \subseteq I$ $\quad$ $I \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \in I$$

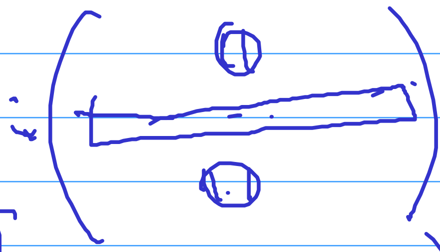




$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad n^2$$



$\in I$



$I = M_n(R)$ ∎

Direct sums of rings

Direct products of groups

$G_1 \quad G_2$

$G_1 \times G_2 = \{(x_1, x_2) \mid x_1 \in G_1, x_2 \in G_2\}$

$(x_1, x_2)\cdot(y_1, y_2) \overset{df}{=} (x_1 y_1, x_2 y_2)$

$x_1, y_1 \in G_1 \qquad ((x_1, x_2)\cdot(y_1, y_2))(z_1, z_2) =$

$x_2, y_2 \in G_2 \qquad = ((x_1 y_1) z_1, (x_2 y_2) z_2)$

$\qquad\qquad\qquad (x_1(y_1 z_1), x_2(y_2 z_2))$

$(1, 1)\cdot(x_1, x_2) = (x_1, x_2)$

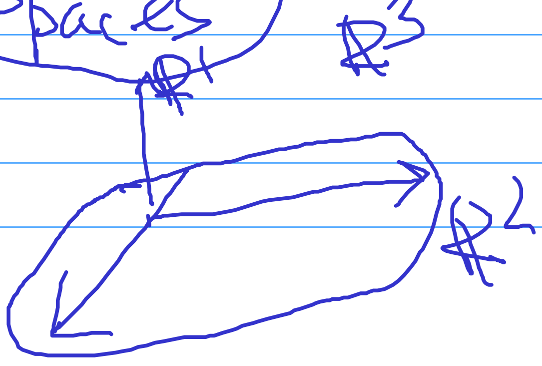
$(x_1, x_2)^{-1} = (x_1^{-1}, x_2^{-1})$

direct sum of abelian groups

$C_n \quad \mathbb{Z}_n \quad \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots$

direct sum of vector spaces

$V_1 \oplus V_2$



$\mathbb{R}^2 \oplus \mathbb{R}^1 \cong \mathbb{R}^3$

$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \cong \mathbb{R}^3$

$(G_1 \times G_2) \times G_3 \simeq G_1 \times (G_2 \times G_3)$

$((x,y),z) \longleftrightarrow (x,(y,z))$

$G_1 \times G_2 \times G_3 \times \dots \times G_n$

$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 y_1, \dots)$

$(1,1,1)$ $\quad (x_1, x_2, x_3 \dots)$

$\prod_{i \in I} G_i \qquad \bigoplus_{i \in I} G_i$

$\coprod \qquad \coprod$