EQUIVALENCE OF METRICS

Theorem 1 (Munkres, Lemma 13.3). *Let Ω and Ω' be two topologies on a set X, with bases Σ and Σ' respectively. Then Ω' is finer than Ω (i.e., $\Omega \subseteq \Omega'$) if and only if for each $B \in \Sigma$, and each $x \in B$, there is $B' \in \Sigma'$ such that $x \in B' \subseteq B$.*

Proof. $\Rightarrow$: Suppose Ω' is finer than Ω. Let $x \in B \in \Sigma$. Then $B \in \Sigma \subseteq \Omega \subseteq \Omega'$, hence $B = \bigcup_{i \in I} B'_i$ for some $B'_i \in \Sigma'$. Since $x \in \bigcup_{i \in I} B'_i$, there is $i_0 \in I$ such that $x \in B'_{i_0}$, and then $x \in B'_{i_0} \subseteq B$ with $B'_{i_0} \in \Sigma'$.

$\Leftarrow$: Let $U \in \Omega$. Take arbitrary $x \in U$. Since U is a union of elements of Σ, there is $B_x \in \Sigma$ such that $x \in B_x \subseteq U$. Then there is $B'_x \in \Sigma'$ such that $x \in B'_x \subseteq B_x$. Then

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B'_x \subseteq \bigcup_{x \in U} B_x \subseteq U,$$

i.e., $U = \bigcup_{x \in U} B'_x$, and $U \in \Omega'$. Consequently, $\Omega \subseteq \Omega'$. ☐

Definition 1. Two metric spaces (X, ρ_1) and (X, ρ_2) are called *topologically equivalent*, if they induce the same topology on X.

Lemma 1 (Munkres, Lemma 20.2). *Let ρ and ρ' be two metrics on the set X. Then the topology induced by ρ' is finer than the topology induced by ρ, if and only if for each $x \in X$, and each $\varepsilon > 0$, there is $\delta > 0$ such that $B^{\rho'}_{\delta}(x) \subseteq B^{\rho}_{\varepsilon}(x)$.*

Proof. $\Rightarrow$: Suppose the topology induced by ρ' is finer than the topology induced by ρ. Fix $x \in X$ and $\varepsilon > 0$. By Theorem 1, there is a ball B' with respect to the metric ρ', such that $x \in B' \subseteq B^{\rho}_{\varepsilon}(x)$. Inside B' we can find a ball $B^{\rho'}_{\delta}(x)$ with the center x.

$\Leftarrow$: Let B is a ball with respect to the metric ρ, and $x \in B$. Then there is a ball $B^{\rho}_{\varepsilon}(x) \subseteq B$ with the center x. Then there is $\delta > 0$ such that $B^{\rho'}_{\delta}(x) \subseteq B^{\rho}_{\varepsilon}(x)$. By Theorem 1, the topology induced by ρ' is finer than the topology induced by ρ. ☐

Definition 2. Two metric spaces (X, ρ_1) and (X, ρ_2) are called *metrically equivalent*, if there are $c, C > 0$ such that

$$c\rho_1(x,y) \leq \rho_2(x,y) \leq C\rho_1(x,y) \tag{1}$$

for any $x, y \in X$.

Theorem 2. *Metric equivalence is an equivalence relation.*

Proof. The condition (1) is equivalent to

$$\frac{1}{C}\rho_2(x,y) \leq \rho_1(x,y) \leq \frac{1}{c}\rho_2(x,y)$$

for any $x, y \in X$, what implies symmetricity. The reflexivity and transitivity are obvious. ☐

Theorem 3. *If two metric spaces are metrically equivalent, then they are topologically equivalent.*

Proof. The condition (1) implies $B^{\rho_1}_{\frac{\varepsilon}{C}}(x) \subseteq B^{\rho_2}_{\varepsilon}(x)$ for any $x \in X$ and $\varepsilon > 0$. By Lemma 1, the topology induced by ρ_1 is finer than the topology induced by ρ_2. By Theorem 2, we can interchange ρ_1 and ρ_2, and then the topology induced by ρ_2 is finer than the topology induced by ρ_1. Hence these two topologies are equivalent. ☐

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