# SIMPLICITY OF QUATERNIONS

**Theorem.** *The quaternion algebra is simple.*

Let us denote the quaternion algebra over a field $K$ by $\mathbb{H}(K)$.

*First proof.* Extend the base field $K$ to the algebraic closure $\overline{K}$, and consider the quaternion algebra over $\overline{K}$: $\mathbb{H}(\overline{K}) = \mathbb{H}(K) \otimes_K \overline{K}$. We know that $\mathbb{H}(\overline{K}) \simeq M_2(\overline{K})$, and the latter algebra is simple. Hence $\mathbb{H}(K)$ is simple. □

*Second proof.* Let $I$ be a nonzero ideal of $\mathbb{H}(K)$. Take a nonzero element $\alpha 1 + \beta i + \gamma j + \delta k \in I$, where $\alpha, \beta, \gamma, \delta \in K$. Multiply this element on $i$ from the right and from the left we get, respectively, two elements lying in $I$:

$$-\beta 1 + \alpha i + \delta j - \gamma k$$
$$-\beta 1 + \alpha i - \delta j + \gamma k.$$

Taking the sum and the difference of these two elements and dividing by 2 (at this point we assume that characteristic of $K$ is not 2; the proof in characteristic 2 can go along the same lines, but would be more cumbersome), we get, respectively, $-\beta 1 + \alpha i \in I$ and $\delta j - \gamma k \in I$. Since not all of $\alpha, \beta, \gamma, \delta$ are zero, at least one of those two elements is not zero.

Case 1. $I$ contains a nonzero element of the form $\lambda j + \mu k$ for some $\lambda, \mu \in K$.

Case 1a. $\lambda \neq 0$. Multiplying the element $\lambda j + \mu k$ on $j$ from the right and from the left, and adding the results of multiplications, we get $-2\lambda 1 \in I$, hence $1 \in I$ and $I = \mathbb{H}(K)$.

Case 1b. $\lambda = 0$. Then $\mu \neq 0$, and $k \in I$. Then $k^2 = -1 \in I$, and again $I = \mathbb{H}(K)$.

Case 2. $I$ contains a nonzero element of the form $\lambda 1 + \mu i$ for some $\lambda, \mu \in K$. Multiplying this element on $j$ from the right, we get a nonzero element $\lambda j + \mu k$, i.e., we are in Case 1.

A more systematic approach would consist of multiplying the initial element $\alpha 1 + \beta i + \gamma j + \delta k$ on all $i$, $j$, $k$ from the right and from the left, and collecting all the results:

$$-\beta 1 + \alpha i + \delta j - \gamma k$$
$$-\beta 1 + \alpha i - \delta j + \gamma k$$
$$-\gamma 1 - \delta i + \alpha j + \beta k$$
$$-\gamma 1 + \delta i + \alpha j - \beta k$$
$$-\delta 1 + \gamma i - \beta j + \alpha k$$
$$-\delta 1 - \gamma i + \beta j + \alpha k$$

The linear span of the initial element and those 6 elements lies in $I$ (actually, coincides with $I$) and its dimension is equal to the rank of the $4 \times 7$ matrix of the corresponding coefficients:

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\beta & \alpha & \delta & -\gamma \\ -\beta & \alpha & -\delta & \gamma \\ -\gamma & -\delta & \alpha & \beta \\ -\gamma & \delta & \alpha & -\beta \\ -\delta & \gamma & -\beta & \alpha \\ -\delta & -\gamma & \beta & \alpha \end{pmatrix}$$

One can check then (for example, using some computer algebra system) that if $\alpha, \beta, \gamma, \delta$ are not all zero, the rank of this matrix is always 4. This approach is more laborious, but also more elegant. □

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