COHOMOLOGY OF ALGEBRAS.
SYNOPSIS OF LECTURES AT SCNU, MAY-JUNE 2010

PASHA ZUSMANOVICH

The (largely unachieved, I presume) goals of these lectures were: first, to give as elementary as possible introduction to various cohomology theories, with emphasis on (low-dimensional) cohomology of Lie algebras; second, to introduce some research problems in this and related areas; third, to demonstrate relationship of some homological questions with Gröbner–Shirshov bases (GS bases in the sequel).

In most of the cases, proofs were not given. Instead, some simple examples were considered, and ideas behind a proof were (tried to be) explained in such a way that it should become just a matter of straightforward (but often cumbersome and not performed in the class) calculations.

Lecture 1

1.1. **A five-minute introduction to homological algebra.** The subject of homological algebra. Definition of cochain complex and its cohomology. Example: simplicial complex in geometric and algebraic terms. Main equation of homological algebra: $d^2 = 0$ and its geometric meaning: "boundary of boundary is zero".

Example: cohomology of complex with zero differential.

Cochains, cocycles and coboundaries. Map of complexes induces map of their cohomology.

Recommended books: [CE], [HS], [Lo].

1.2. **Lie algebra cohomology.** Definition of Chevalley–Eilenberg complex for cohomology of a Lie algebra with coefficients in a module (explicit formula for differential).

Why cohomology of Lie algebras is interesting? Because it provides connection with different areas of mathematics and physics, and gives a unified language for various problems arising in the structure theory of Lie algebras.

Low-dimensional interpretations of Lie algebra cohomology. $H^0(L,M)$ is a submodule of invariants. $H^1(L,K) \simeq (L/[L,L])^*$. $H^1(L,L)$ are outer derivations. $H^2(L,K)$ are central extensions and, more generally, $H^2(L,M)$ are abelian extension.

Why extensions of Lie, associative, and other classes of algebras are important? Because they provide a tool to build general algebras out of semisimple and solvable (nilpotent) ones. For finite dimensional Lie algebras in characteristic zero, we have the Wedderburn–Malcev theorem about splitting of arbitrary Lie algebra into its semisimple part and radical, and for associative algebras, we have the similar Wedderburn (?) theorem.

Besides, central extensions of Lie algebras are very much loved by physicists: in classical mechanics, particles are moving in a phase space, on which groups – and through them, Lie algebras – act. In quantum mechanics, phases are unobservable, and groups and corresponding Lie algebras act "up to a phase", i.e. we get projective representations. But projective representations of a Lie algebra are exactly the same as representations of its central extension.

That is why, when devising new theories and their symmetries, physicists (almost) always require that the corresponding Lie algebras should have a nontrivial central extension.

Recommended book: [F].

Date: last minor revision September 16, 2019.

Lecture 2

2.1. **Some examples.** Examples of computation of Lie algebra cohomology: $H^*(L, K)$ for L abelian and two-dimensional nonabelian.

Examples of central extensions: the simplest Kac–Moody ($sl(2) \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}z$) and Virasoro algebra.

2.2. **Deformations.** Deformations of Lie algebras. Massey bracket. Interpretation of $H^2(L, L)$ as infinitesimal deformations and of $H^3(L, L)$ as obstructions to prolongability of infinitesimal deformations. Rigid algebras. A sufficient condition for rigidity is $H^2(L, L) = 0$. Example of a rigid Lie algebra: $sl(2)$. Example of a (highly) non-rigid Lie algebra: abelian Lie algebra.

Filtered and associated graded algebras. Filtered deformations. Infinitesimal filtered deformations $H^2_+(L, L)$ – cocycles preserving filtration arising from grading.

Literature: [GS] (if the latter is unavailable, then the original papers [G] will be a good substitute).

2.3. **Modular Lie algebras.** Filtered (and more general) deformations are ubiquitous in the structure theory of finite-dimensional simple modular Lie algebras. Brief overview of state of the matters with this theory: characteristic zero case was settled in the period from the turn of the XIX-XX centuries till 1940s in the classical works of Killing, H. Cartan, Dynkin and others. First example of a simple modular Lie algebra not having a characteristic zero counterpart by Witt at 1940. Chaotic period in 1950s: more and more exotic examples of algebras, not any plausible conjecture about classification in sight, even not clear what are isomorphism classes of all examples found. Breakthrough by Kostrikin–Shafarevich in 1960s: similarity with infinite-dimensional Lie algebras of Cartan type, conjecture that any simple Lie algebra is either of classical or Cartan type. Long period, spanned thousands of pages and efforts of dozens of people up to beginning of 2000s: classification achieved for $p > 3$ (Strade, Premet). For $p = 2, 3$, the situation till recently was somewhat similar to the chaotic situation with generic p in 1950s: lot of discovered and rediscovered examples (Leites, Grishkov, Elduque – see arXiv), but recently Leites came up with some adjustment of the Kostrikin–Shafarevich conjecture for small p.

Deformations help to get new examples of algebras and serve as an organizational tool.

2.4. Examples of cohomology and deformations of some current Lie algebras.

2.4.1. *An example of deformation.* Example of deformation from [Z1], arising when solving some structural modular problem: $W_1(1) \otimes A$, where $W_1(1)$ is a p-dimensional Witt algebra, and A is an arbitrary associative commutative algebra, plus cocycle:

$$\Phi_D(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p-2} \otimes (aD(b) - bD(a)), & i = j = -1 \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $D \in Der(A)$. Note that the Massey bracket of Φ_D with itself is zero, hence it is prolonged trivially to the global deformation.

2.4.2. *Examples where $H^2_+(L, L)$ is smaller than $H^2(L, L)$.*

$$H^2_+(sl(2) \otimes A, sl(2) \otimes A) = 0$$
$$H^2(sl(2) \otimes A, sl(2) \otimes A) \simeq Har^2(A, A).$$

That means, in particular, that as a graded algebra, $sl(2) \otimes A$ is rigid, and all its (not filtered) deformations are determined by deformations of A in the class of associative commutative algebras.

$Har^2(A, A)$ is defined as the quotient of the space of Harrison 2-cocycles, which are symmetric bilinear maps $f : A \times A \to A$ satisfying

$$af(b, c) - f(ab, c) + f(a, bc) - f(a, b)c = 0,$$

by the space of Harrison 2-coboundaries, which are bilinear maps of the form

$$f(a,b) = ag(b) - g(ab) + g(a)b$$

for some linear map $g : A \to A$.

(2)
$$H^2_+(W_1(1) \otimes A, W_1(1) \otimes A) \simeq Der(A)$$
(3)
$$H^2(W_1(1) \otimes A, W_1(1) \otimes A) \simeq \Big(H^2(W_1(1), W_1(1)) \otimes A\Big) \oplus Der(A) \oplus Der(A) \oplus Har^2(A,A).$$

The cohomology classes in (2) are spanned by cocycles of kind (1).

Filtered deformations of $W_1(1) \otimes A$ are described in [Z1]: they are, essentially, of kind (1).

Question. Describe *all* deformations of $W_1(1) \otimes A$ in terms of A.

Why this is interesting? First, this will give new relationship between various class of Lie algebras. Second, specializing to the case $A =$ reduced polynomial algebra, and adding "tail" of derivations, this will provide first examples in the literature of full description of deformations of modular semisimple Lie algebras (deformations of various *simple* Lie algebras were considered before).

Lecture 3

3.1. **Some generalities on cohomology of current Lie algebras.** The appearance of the rightmost and leftmost summands in (3) is not accidental. If we are interested in cohomology of current Lie algebras $L \otimes A$, say, of the second cohomology, we always have cocycles of the following two kinds:

$$\Phi(x \otimes a, y \otimes b) = \varphi(x,y) \otimes abu$$

for $\varphi : L \times L \to L$ and $u \in A$, and

$$\Phi(x \otimes a, y \otimes b) = [x,y] \otimes f(a,b),$$

where $f : A \times A \to A$.

It is easy to see that they will be 2-cocycles on $L \otimes A$ if and only if φ is a 2-cocycle on L, and f is a Harrison 2-cocycle, respectively. Hence, for any Lie algebra L, we have

$$H^2(L \otimes A, L \otimes A) \supseteq H^2(L,L) \otimes A + Har^2(A,A).$$

This generalizes to the higher degree cocycles.

That is one of the niceties of cohomology of current Lie algebras – it intertwins different cohomology theories.

Other reason why current Lie algebras and their cohomology are interesting: current Lie algebras appear in many contexts – for example, Kac-Moody algebras and modular semisimple Lie algebras (according to the classical Block's theorem about the structure of modular semisimple Lie algebras), are certain extensions of particular current Lie algebras.

3.2. **Examples of deformations of associative commutative algebras.**

3.2.1. $K[x]/(x^2)$ deforms to $K[x]/(x^2 - t1)$, but if the ground field K is quadratically closed, then the deformed algebra is isomorphic to the initial one.

3.2.2. *Universal enveloping algebra.* PBW theorem: an algebra associated with the universal enveloping algebra $U(L)$, is an algebra of polynomials $S(L)$, so, in a sense, $U(L)$ can be considered as a filtered deformation of $S(L)$, though, probably, this is not very rigorous, as algebras are infinite-dimensional and, generally, we should deal with infinite sums. This can be turned, though, to a perfectly rigorous statement by considering a p-analog: restricted universal envelope is a filtered deformation of the reduced polynomial algebra $K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$.

It is interesting to note that the shortest proof of PBW theorem is via GS bases.

3.3. **Cohomology of associative algebras.** Definition of Hochschild cohomology of an associative algebra with coefficient in a bimodule (explicit formula for differential).

The interpretations of $H^0(A, M)$, $H^1(A, A)$, $H^2(A, M)$, $H^{2,3}(A, A)$ are completely similar to those in the case of Lie algebras.

Question. In [Che], Chen essentially described $H^2(A, M)$ in terms of GS basis. First: how does it related to [A]? Second: in particular, we have a description of infinitesimal deformation $H^2(A, A)$ in terms of GS basis of A. Could it be extended to description of *all* deformations of A in terms of its GS basis? Or at least of filtered deformations? The latter question can be reformulated in terms avoiding any homological language: what are relations of GS bases of a filtered algebra and its associated graded algebra? (Something concerning the latter question is contained in [Li]).

3.4. **Cohomology of associative commutative algebras.** Harrison cohomology – defined explicitly in degrees 0, 1 (coincides with Hochschild), and 2 (symmetrized Hochschild cocycles).

3.5. **A bit more of homological algebra.** Definition of a free resolution of a module (always can be constructed explicitly). Derived functors. Chevalley–Eilenberg cohomology $H^n(L, M)$ is the derived functor of $M \mapsto Hom_{U(L)}(K, M) \simeq M^L$, and Hochschild cohomology $H^n(A, M)$ is the derived functor of $M \mapsto Hom_{A^e = A \otimes A^{op}}(A, M)$.

Lecture 4

4.1. **Examples of free resolutions.**

4.1.1. *Example from* [Lo]. Let A be an associative commutative algebra, and $a, b \in A$ such that $Ann(a) = Ab$ and $Ann(b) = Aa$. Then

$$\dots \xrightarrow{R_a} A \xrightarrow{R_b} A \xrightarrow{R_a} A \xrightarrow{R_b} \dots \xrightarrow{R_a} A \to A/Aa \to 0,$$

where R_a is a multiplication by a, is a free resolution of the A-module A/Aa.

This can be further specialized to the case $A = K[x]/(x^n - 1)$, $a = 1 - x$, and $b = 1 + x + x^2 + \dots + x^{n-1}$.

4.1.2. *Example from* [Z1]. Consider the following free resolution of the $K[x]/(x^n)$-module $K[x]/(x^n)$:

$$\dots \xrightarrow{d_2} K[x]/(x^n) \otimes K[x]/(x^n) \xrightarrow{d_1} K[x]/(x^n) \otimes K[x]/(x^n) \xrightarrow{m} K[x]/(x^n) \to 0,$$

where m is multiplication, and

$$d_i(a \otimes b) = \begin{cases} a \otimes xb - ax \otimes b, & i \text{ even} \\ \sum\limits_{k=0}^{n-1} ax^k \otimes x^{n-1-k} b, & i \text{ odd.} \end{cases}$$

Then, applying the functor $Hom_{K[x]/(x^n) \otimes K[x]/(x^n)}(K[x]/(x^n), \cdot)$ to this resolution, we get a complex computing the Hochschild cohomology $H^i(K[x]/(x^n), K[x]/(x^n))$:

$$0 \to K[x]/(x^n) \xrightarrow{id} K[x]/(x^n) \xrightarrow{0} K[x]/(x^n) \xrightarrow{0} \cdots .$$

Hence $H^i(K[x]/(x^n), K[x]/(x^n)) \simeq K[x]/(x^n)$ for any i.

4.2. **Low-dimensional interpretation of cohomology: Lie bialgebras.** Notion of coalgebra and bialgebra.

Lie coalgebra satisfies "co-anticommutativity" (which means that the comultiplication take values in $L \wedge L$, i.e. $\Delta : L \to L \wedge L$) and "co-Jacobi identity", which is nothing but $\Delta \in Z^1(L, L \wedge L)$. Lie bialgebra, additionally, satisfies the "compatibility condition": the induced map $\Delta^* : L^* \wedge L^* \to L^*$ is a Lie algebra (i.e. satisfies the Jacobi identity).

Accordingly, we have coboundary and non-coboundary Lie bialgebra structures, depending whether Δ is a coboundary or not. For classical Lie algebras $\mathfrak{g}$, $H^1(\mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g}) = 0$, so each Lie coalgebra structure is coboundary, and the compatibility condition for bialgebras leads to CYBE (Belavin, Drinfeld, et al.)

Examples of Lie bialgebras:

(i) two-dimensional nonabelian $\langle x, y \,|\, [x, y] = x \rangle$ with $\Delta(x) = x \wedge y$, $\Delta(y) = 0$.
(ii) $sl(2) = \langle e_-, h, e_+ \,|\, [e_-, h] = -e_-, [e_+, h] = e_+, [e_-, e_+] = h \rangle$ with $\Delta(e_\pm) = e_\pm \wedge h$, $\Delta(h) = 0$.

More generally, there is some work(s?) where it is proved that any finite-dimensional Lie algebra over an algebraically closed field of characteristic zero has a nontrivial Lie bialgebra structure, along the following way: every nonabelian Lie algebra is either nilpotent, and hence contains a 3-dimensional nilpotent (Heisenberg) subalgebra, or contains a non-nilpotent element and hence contains a 2-dimensional nonabelian subalgebra. By choosing complimentary basis in a certain way, one can induce the bialgebra structure on the whole algebra from the respective small subalgebras.

4.3. **Low-dimensional interpretation of homology: generators and relations of nilpotent (and close to them) Lie algebras.** Definition of the homology complex for Lie algebras.

For (generalized) nilpotent Lie algebras, $H_1(L, K)$ is interpreted as the linear space spanned by generators, and $H_2(L, K)$ – as a linear space spanned by relations.

Of course, this is not true in general – take, for example, the classical simple Lie algebras $\mathfrak{g}$ for which both cohomology are zero. But still it may be useful even for such algebras, due to triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ ($\mathfrak{h}$ is a Cartan subalgebra, and $\mathfrak{n}_\pm$ are "lower", respectively "upper", triangular parts. Relations between $\mathfrak{h}$ and $\mathfrak{n}_\pm$, and between $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are easy, so the question about relations in such algebra can be reduced to relations between elements of $\mathfrak{n}_-$ and $\mathfrak{n}_+$. The latter algebras are nilpotent and for them the homological interpretation above is true.

Generalizing this, we may consider a triangular decomposition of affine Kac-Moody Lie algebras, which gives a seemingly shortest possible (and homological) proof of analogs of the Serre defining relations in such algebras (cf. [Z3]).

In classical Lie algebras, GS bases corresponding to Serre defining relations were computed by Bokut and Klein in 1990s.

Question. Compute GS bases for known finite-dimensional simple Lie algebras in $p = 2, 3$ (see the recent arXiv papers of Leites and Elduque and references therein).

This, possibly, could help in making order in the current "zoo" of such Lie algebras.

4.4. **Low-dimensional interpretation of cohomology:** H^3**.** There is another one, recent, interpretation of the third cohomology of Lie algebras. It appeared in works of Baez & Co. about so-called 2-Lie algebras, which are categorical generalizations of ordinary Lie algebras, and are related with modern physical theories.

Lecture 5

5.1. **Why computation of cohomology, and, in particular, Lie algebra cohomology, does not reduce to a mere linear algebra?** First, the dimensions of appropriate linear spaces of cochains grow prohibitively fast and are not amenable to direct computations and general linear-algebraic technique, either by hand or by computer.

Second, we want to solve the corresponding linear systems exactly, and not numerically, and sometimes over some esoteric fields, what makes the arsenal from numerical linear algebra not applicable directly. There are, though, some interesting connection between cohomology and numerical linear algebra to explore.

Question. To devise algorithms/write computer program for computation of Lie algebra cohomology, which takes into account the sparsity of the differential.

In numerical linear algebra, there is a big arsenal of methods to work with sparse matrices, and though they are not directly applicable to cohomological computations, some ideas and methods from there could be very relevant.

Third, we often have some additional structure on a Lie algebra and/or its module (some action, for example), what induces structure on cohomology, and often is this additional structure what allows to make interesting connections to other branches of mathematics and physics.

5.2. **A tool for computation of Lie algebra cohomology: a cohomological long exact sequence associated with a short exact sequence of modules.** This is a general homological algebra thing: the short exact sequence of modules leads to the short exact sequence of cochain complexes, which, via the Snake Lemma, leads to the long exact sequence of their cohomology. See "Snake Lemma" in wikipedia.

The same is true for Hochschild cohomology.

5.3. **A tool for computation of Lie algebra cohomology: triviality of cohomology under the torus action.** Exposition after [F].

Examples of application:

(i) $H^2(sl(2), K)$;
(ii) $H^2(W_1, K)$, where W_1 is two-sided infinite-dimensional Witt algebra.

Lecture 6

The first part of this lecture was somewhat unusual. I deviated from the usual course and presented a few half-baked (or, rather, not baked at all) ideas related to GS bases.

6.1. **GS bases of toroidal, Krichever–Novikov, etc., algebras.** Would it be possible to write GS basis of a general current Lie algebra $L \otimes A$ from GS bases of L and A, similar to what is done for the tensor product of two associative algebras? If we will be able to do that, specializing L to classical simple, and A to various polynomial-like algebras, we may infer GS bases of various generalizations of affine nontwisted Kac–Moody: toroidal, Krichever–Novikov, etc. (and may be rediscover, or have a new insight on very complex-looking GS bases of Kac–Moody by Poroshenko [Po1, Po2]).

This can be also thought about along the following lines: a GS basis for a Lie algebra *over an associative commutative algebra* A, i.e. $L \otimes_A A$, can be constructed from the corresponding data for L_K and A_K, but we need to do it *over* K. Can we, in general, to infer a GS base of a Lie algebra over a field (or, more generally, a ring) K if GS basis of L over A and of A over K are known?

6.2. **Algebras represented as the sum of subalgebras, and Rota–Baxter algebras.** There is a circle of questions related to decomposition of a Lie (or associative) algebra into the sum of two its subalgebras: $L = A + B$ (the sum is understood as the vector space sum and is not necessarily direct). Two open problems:

(i) In the category of Lie algebras, let A and B are nilpotent. Is it true that L is solvable? (For finite-dimensional Lie algebras, this is almost trivial – modulo structure theory – in characteristic zero, and is the solved Kegel–Kostrikin problem in characteristic p. For infinite-dimensional Lie algebras, it is known to be true in few cases, under various finiteness assumptions. The general case is open).

(ii) In the category of associative algebras, let A and B be PI. Is it true that L is PI? (It is known to be true in a lot of particular cases, the general case is open).

Assume that the sum (of vector spaces) is direct: $L = A \oplus B$. Then, for a projection operator $P : L \to A$, we have

$$[P(x), P(y)] - P([P(x), y] + [x, P(y)]) + P([x, y]) = 0 \tag{4}$$

in the Lie algebras case, and a similar condition in the associative case (and, of course, $P^2 = P$). But, in the associative case, this is exactly the Rota–Baxter operator for $\lambda = -1$! (or $\lambda = 1$?). Then, couldn't we talk about decomposition into the vector space direct sum of subalgebras with given conditions, in the language of Rota–Baxter algebras, by considering them as quotient of a free Rota–Baxter algebra? If so, couldn't we apply the knowledge of the structure of free Rota–Baxter algebras? (Of course, this is for associative case, Lie analogs of Rota–Baxter algebras have to be developed).

Further, there are lot of conditions similar to (4) appearing in different contexts (Nijenhuis operator, R-matrix, Kähler structure, etc.). Wouldn't it be beneficial to consider the analogs of Rota–Baxter algebras for them and build the corresponding GS-base theory?

6.3. **Girth.** Could we utilize the GS theory for questions related to girth? (For example, determine, whether a given group or algebra has infinite girth or not).

6.4. **Grassmann envelope.** Would it be interesting to relate GS bases of a (Lie) superalgebra and its Grassmann envelope?

6.5. **Databases.** Develop a theory of GS bases for databases (for example, in a sense of [Pl]). This could be of interest to computer scientists.

Then we proceeded with the usual course.

6.6. **Spectral sequences in general.** Invented during WWII by Leray in the German concentration camp. Have an unjust reputation of being obscure and difficult subject.

An idea of spectral sequence by example of filtered and associated graded complex. The idea of successive approximation: each term is cohomology of the previous term approaching the cohomology in question.

Geometrical representation of each term of a spectral sequence on the Cartesian plane.

Literature: [Cho].

6.7. **A tool for computation of Lie algebra cohomology: Hochschild–Serre spectral sequence.** Exposition according to [F].

Example of application:

$$H^*(L_1 \oplus L_2, K) \simeq H^*(L_1, K) \otimes H^*(L_2, K). \tag{5}$$

Widely used to compute cohomology of central and non-central extensions, semidirect products, etc. Often, in "real life", degenerates at E_2.

7.1. Künneth theorem. Tensor product of two complexes. Why we have $(-1)^{deg x}$ in the formula

$$\delta(x \otimes y) = d(x) \otimes y + (-1)^{deg x} x \otimes d'(y)$$

for differential? Because without it, we will not have $\delta^2 = 0$ (check for the low-dimensional case). This is a manifestation of some inherent presence of super-mathematics in cohomology theories.

Cohomology of the tensor product of complexes is isomorphic to the tensor product of cohomology.

7.2. Corollaries of the Künneth theorem. The shuffle map

$$sh_{ij} : C^i(A, A) \otimes C^j(B, B) \to C^{i+j}(A \otimes B, A \otimes B)$$

$$(f \times g)(a_1 \otimes b_1, \dots, a_{i+j} \otimes b_{i+j}) = \sum_{\substack{\sigma \in S_{i+j} \\ \sigma(1) < \sigma(2) < \dots < \sigma(i) \\ \sigma(i+1) < \sigma(i+2) < \dots < \sigma(i+j)}} (-1)^{\sigma} f(a_{\sigma(1)}, \dots, a_{\sigma(i)}) \otimes g(b_{\sigma(i+1)}, \dots, b_{\sigma(i+j)})$$

induces the isomorphism of the tensor product of Hochschild complexes $C^*(A, A) \otimes C^*(B, B)$ with the Hochschild complex $C^*(A \otimes B, A \otimes B)$. Hence, by Künneth theorem,

$$H^*(A \otimes B, A \otimes B) \simeq H^*(A, A) \otimes H^*(B, B)$$

for any two associative (unital?) algebras A and B.

As another corollary, we get again the formula (5) for cohomology of the direct sum of Lie algebras. This is related to the fact that $U(L_1 \oplus L_2) \simeq U(L_1) \otimes U(L_2)$, if we look on the Lie algebra cohomology from the viewpoint of derived functors.

7.3. Cohomology of current Lie algebras. But if we look on the tensor product in the category of Lie algebras, i.e., on current Lie algebras $L \otimes A$, there is nothing like Künneth theorem analog, and, generally, their cohomology is difficult to compute.

Explanation of the linear-algebraic technique from [Z2] and [Z3] for computation of (some) of the low-dimensional cohomology of current Lie algebras, derivation of the formula

$$H^2(L \otimes A, K) \simeq \Big(H^2(L) \otimes A^*\Big) \oplus \Big(B(L) \otimes HC^1(A)\Big)$$

for L perfect and A unital, where $B(L)$ is the space of symmetric bilinear invariant forms, and $HC^1(A)$ is the first-order cyclic cohomology, i.e., the space of all skew-symmetric bilinear maps $\alpha : A \times A \to K$ such that $\alpha(ab, c) + \alpha(ca, b) + \alpha(bc, a) = 0$.

Reference for cyclic cohomology: [Lo].

This generalizes the famous central extension appearing in affine Kac-Moody Lie algebras.

Again, we see that cohomology of current Lie algebras intertwins various other cohomology theories.

Derivation, in a similar (and simpler) way, of the formula

$$B(L \otimes A) \simeq B(L) \otimes A^*$$

which generalizes Lemma 2 from [MSZ].

Question. Compute in the same way $H^3(L \otimes A, K)$.

This should be the last cohomology degree fully amenable to such sort of computations. [H] treats some particular result in that direction (third cohomology of some subcomplex of the Chevalley–Eilenberg complex of $L \otimes A$, over a field of characteristic zero).

The same simple technique should be applicable to description of many other structures, besides cohomology, on current Lie algebras: Poisson structures, Lie-admissible structures, Hom-Lie algebras, etc, etc., which should generalize and unify many existing results about Kac–Moody, toroidal, and similar algebras.

7.4. **A possible approach to computation of higher cohomology of current Lie algebras.** How this can be generalized to cohomology in degree > 2? We should take more symmetries into account, besides total skew-symmetry and total symmetry.

Young tableaux, Young diagrams, Young symmetrizers, Cauchy formula. The graph of Young diagrams as pictorial representation of current Lie algebra cohomology. Vanishing patterns in it (according to [Z2]).

7.5. **Cocycles in "general position".** The crucial fact in our computations of low-dimensional cohomology of current Lie algebras is this: if S, S' is a pair of linear operators with the same source and target, and T, T' is another pair of such operators, then

$$\begin{aligned} Ker(S \otimes T) \cap Ker(S' \otimes T') = (KerS \cap KerS') \otimes (\text{whole target}) \\ + KerS \otimes KerT' + KerS' \otimes KerT + (\text{whole source}) \otimes (KerT \cap KerT'). \end{aligned}$$

This elementary linear-algebraic statement does not generalize to three or more intersections, and this is one of the main obstacles for further applicability of this method.

There is a similar situation with the known linear-algebraic statement

$$\dim V + \dim W = \dim(V + W) - \dim(V \cap W)$$

for two arbitrary vector spaces V, W.

The notion of general position.

Question. Could we have cocycles "in general position"? (or, rather, cocycles, for which all decomposable consequences of cocycle equation are generated by linear operators in general position with respect to each other). If yes, what part of cohomology that will be? Some nontrivial algebraic geometry, or Gelfand–Ponomarev theory, could be possibly involved.

Lecture 8

8.1. **Operads.** We adopt a primitive view on operads as graded vector spaces glued from multilinear graded components of free algebras in the corresponding varieties of algebras.

For a not primitive view, see [MSS], [Ca] and [S].

Binary quadratic operad. Pairing on the space of quadratic relations. Koszul duality.

Remarkable fact: if A is an algebra over a binary quadratic operad $\mathcal{P}$, and B is an algebra over its dual $\mathcal{P}^!$, then $(A \otimes B)^{(-)}$ with multiplication

$$[a \otimes b, a' \otimes b'] = aa' \otimes bb' - a'a \otimes b'b$$

is a Lie algebra. Utilization of this fact for finding identities of the dual operad from the identities of the original operad.

$Ass^! = Ass$, $Lie^! = Comm$, $Comm^! = Lie$.

Question. Is it possible to write a GS basis for the Lie algebra $(A \otimes B)^{(-)}$, from GS bases of A and B, where A is an algebra over a quadratic binary operad $\mathcal{P}$, and B is an algebra over the Koszul dual operad $\mathcal{P}^!$?

This is a vast generalization of question from §6.1.

Poincaré series of an operad. Calculation of Poincaré series for Ass, $Comm$ and Lie.

Koszulity. Ginzburg–Kapranov criterion.

Question (Zelmanov). Is it possible to translate Koszulity from the language of operads to the language of universal algebra? (see, for example, [Pi] about such translation in general).

Alternative operad is not Koszul (after [DZ]).

Question. Do the following polynomials have inverses with alternating signs:

(i) $-x + x^2 - \frac{5}{6}x^3 + \frac{1}{2}x^4 - \frac{1}{8}x^5$ (from [DZ]);

(ii) $-x + x^8 - x^{15}$ (from [MR]).

If yes, what combinatorial interpretations that may have?

Both of them are Poincaré series of some operads (first – of the dual alternative one). Surprisingly, such questions seem to be difficult.

8.2. **Dual alternative operad in $p = 3$.**

Question ([DZ]). Prove that in $p = 3$, $\dim Alt^!(n) = 2^n - n$.

Dual alternative algebras in $p = 3$ are exactly associative algebras whose associated Lie algebra is 2-Engel, so this essentially a question about free algebras in such variety of algebras. This seemingly could be solved easily with the GS technique.

Lecture 9

9.1. **Lie algebras of cohomological dimension 1.** Equivalent definitions (in terms of vanishing of the second cohomology, in terms of projective resolutions, in terms of splitting extensions).

The universal property of free Lie algebras immediately implies that free Lie algebras are of cohomological dimension 1.

Question (Bourbaki). Is it true that the converse is true, i.e. Lie algebras of cohomological dimension 1 are free?

Comparison with the group and associative case.

9.2. **Induced module.** Definition. Shapiro's lemma (proved with the help of the Hochschild–Serre spectral sequence, see [F]).

9.3. **Known results and open questions about Lie algebras of cohomological dimension 1.** A subalgebra of a Lie algebra of cohomological dimension 1 is of cohomological dimension 1. A finite dimensional Lie algebra of cohomological dimensional dimension 1 is 1-dimensional. Feldman's result: a 2-generated Lie algebra of cohomological dimension 1 is free. A Lie p-algebra of cohomological dimension 1 is 1-dimensional.

Mikhalev–Umirbaev–Zolotykh example.

This question begs for application of the GS technique via:

(i) Anick resolution;

(ii) A Lie-algebraic analog of Chen's characterization of extensions in terms of GS bases [Che].

References

[A] D.J. Anick, *On the homology of associative algebras*, Trans. Amer. Math. Soc. **296** (1986), 641–659.

[CE] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, 1956.

[Ca] P. Cartier, *What is an operad?*, Surveys of Modern Mathematics. The Independent Univ. of Moscow Seminars (ed. V. Prasolov), LMS Lect. Notes Ser. **321** (2005), 283–291.

[Che] Y. Chen, *Gröbner-Shirshov bases for extensions of algebras*, Algebra Colloq. **16** (2009), 283–292; arXiv:0804.0639.

[Cho] T.W. Chow, *You could have invented spectral sequences*, Notices Amer. Math. Soc. **53** (2006), 15–19.

[DZ] A. Dzhumadil'daev and P. Zusmanovich, *The alternative operad is not Koszul*, Experim. Math., to appear; arXiv:0906.1272.

[F] D.B. Fuchs, *Cohomology of Infinite-Dimensional Lie Algebras*, Consultants Bureau, N.Y., 1986.

[G] M. Gerstenhaber, *On the deformation of rings and algebras. I-IV*, Ann. Math. **79** (1964), 59–103; **84** (1966), 1–19; **88** (1968), 1–34; **99** (1974), 257–276.

[GS] ——— and S.D. Schack, *Algebraic cohomology and deformation theory*, Deformation Theory of Algebras and Structures and Applications (ed. M. Hazewinkel and M. Gerstenhaber), Kluwer, 1988, 11–264.

[H] A. Haddi, *Homologie de degré trois des algèbres de Lie étendues par une algèbre commutative*, Comptes Rendus Acad. Sci. Paris **321** (1995), 965–968.

[HS] P.J. Hilton and U. Stammbach, *A Course in Homological Algebra*, 2nd ed., Springer, 1997.

[Li] H. Li, *Noncommutative Gröbner Bases and Filtered–Graded Transfer*, Lect. Notes Math. **1795** (2002).

[Lo] J.-L. Loday, *Cyclic Homology*, 2nd ed., Springer, 1998.

[MR] M. Markl and E. Remm, *(Non-)Koszulity of operads for n-ary algebras, cohomology and deformations*, arXiv:0907.1505.

[MSS] ———, S. Shnider, and J.D. Stasheff, *Operads in Algebra, Topology and Physics*, AMS, 2002.

[MSZ] F. Montaner, A. Stolin, and E. Zelmanov, *Classification of Lie bialgebras over current algebras*, Selecta Math. **16** (2010), 935–962; arXiv:1001.4824.

[Pi] D.I. Piontkovski, *On the Kurosh problem in varieties of algebras*, J. Math. Sci. **163** (2009), 743–750; arXiv:0905.4748.

[Pl] B.I. Plotkin, *Galois theory of databases*, Lect. Notes Math. **1352** (1988), 147–161.

[Po1] E.N. Poroshenko, *Gröbner-Shirshov bases for Kac-Moody algebras $A_n^{(1)}$ and $B_n^{(1)}$*, Formal Power Series and Algebraic Combinatorics (ed. D. Krob et al.), Springer, 2000, 552–563.

[Po2] ———, *Gröbner-Shirshov bases for the Kac-Moody algebras of the type $A_n^{(1)}$*, Comm. Algebra **30** (2002), 2617–2637.

[S] J. Stasheff, *What is... an operad?*, Notices Amer. Math. Soc. **51** (2004), 630–631.

[Z1] P. Zusmanovich, *Deformations of $W_1(n) \otimes A$ and modular semisimple Lie algebras with a solvable maximal subalgebra*, J. Algebra **268** (2003), 603–635; arXiv:math/0204004.

[Z2] ———, *Low-dimensional cohomology of current Lie algebras and analogs of the Riemann tensor for loop manifolds*, Lin. Algebra Appl. **407** (2005), 71–104; arXiv:math/0302334.

[Z3] ———, *Invariants of Lie algebras extended over commutative algebras without unit*, J. Nonlin. Math. Phys. **17** (2010), Suppl. 1 (special issue in memory of F.A. Berezin), 87–102; arXiv:0901.1395.

Email address: pasha.zusmanovich@gmail.com