COHOMOLOGY OF ALGEBRAS.
SYNOPSIS OF LECTURES AT UNIVERSITY OF ANTANANARIVO, SEPTEMBER 2019

PASHA ZUSMANOVICH

Each lecture lasted 2 (astronomical) hours.

LECTURE 1

1.1. **Refresher: Lie algebras, associative algebras, associative commutative algebras.** Definitions, $A^{(-)}$, basic examples.

1.2. **Derivations.** Lie algebra $Der(A)$ for a (non-associative) algebra A. Inner derivations of a Lie algebra.

1.3. **Definition of cochain complex.** Cochains, cocycles, coboundaries.

1.4. **Definition of Lie algebra cohomology.** Explicit formula for the differential.

1.5. **Interpretations of low-degree Lie algebra cohomology.**
$H^0(L, M) = M^L$; in particular, $H^0(L, L) = Z(L)$
$H^1(L, K) \simeq (L/[L, L])^*$
$H^1(L, L) =$ outer derivations.

LECTURE 2

2.1. **Central extensions.** Equivalence classes of central extensions are described by $H^2(L, K)$.
Importance of central extensions: structure theory, quantum mechanics.

2.2. **Abelian extensions.** Equivalence classes of abelian extension of a Lie algebra L by a module M are described by $H^2(L, M)$.

2.3. **Deformations.** Massey brackets. Infinitesimal deformations are described by $H^2(L, L)$, obstructions to prolongations of deformations lie in $H^3(L, L)$.

2.4. **Elementary examples of cohomology calculations.**
If L is abelian, then $H^n(L, K) = C^n(L, K)$.
Let $L = Kx$ be 1-dimensional. Then $H^0(Kx, M) = Ker(x_M)$ and $H^1(Kx, M) = M/Im(x_M) = Coker(x_M)$.

LECTURE 3

3.1. **Elementary examples of cohomology calculations (continuation).** $H^*(L, K)$ for L 2-dimensional nonabelian and $L = sl(2)$.

3.2. **Killing form.** Invariant symmetric bilinear form on a Lie algebra.

3.3. **Current Lie algebras.** $L \otimes A$ for a Lie algebra L and associative commutative algebra A. Modular semisimple Lie algebras as extensions of a particular kind of current Lie algebras by a "tail" of derivations.

3.4. **Operads. Koszul duality.** $(A \otimes B)^{(-)}$ where A, B are algebras over Koszul dual operads is a Lie algebra.

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3.5. **Kac-Moody algebras.** Example of a central extension:

$$sl(2) \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}t\frac{t}{dt} + \mathbb{C}z$$

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + B(x, y) Res_{-1}(f\frac{dg}{dt})z$$

3.6. **Filtered deformations.** Graded Lie algebra associated to a filtered Lie algebra. $H^2_+(L, L)$ classifies infinitesimal filtered deformations.

LECTURE 4

4.1. **Gradings.** Grading on cohomology inherited from a grading on an algebra and on a module. Invariance of cohomology with respect to a torus action (= concentration of cohomology in the zero degree).

4.2. **Examples of computation of cohomology using invariance with respect to a torus action.**
$H^2(W, K)$ for infinite-dimensional Witt algebra W.
$H^2(sl(2) \otimes A, K) \simeq HC^1(A)$.

LECTURE 5

5.1. **Examples of computation of cohomology using invariance with respect to a torus action (continuation).**
$H^2(sl(2) \otimes A, sl(2) \otimes A) \simeq Har^2(A, A)$.

5.2. **Hochschild cohomology.** Bimodules. Explicit formula for the differential.

5.3. **Interpretations of low-degree Hochschild cohomology.** Similar to Chevalley–Eilenberg cohomology:
$H^0(A, M) = M^A = \{m \in M \mid a \bullet m = m \bullet a$ for any $a \in A\}$; in particular, $H^0(A, A) = Z(A)$.
$H^1(A, A)$ = outer derivations.
$H^2(A, M)$ classifies square-zero extension of A by M.
$H^2(A, A)$ classifies infinitesimal deformations of A, and obstructions to prolongations of deformations lie in $H^3(A, A)$.

LECTURE 6

6.1. **Harrison cohomology.** The beginning of the Hochschild complex with additional symmetry condition on cochains gives Harrison complex. Low-degree interpretation of the Harrison cohomology is the same as in the Hochschild (associative) case.

6.2. **Three graces.** Associative, associative commutative and Lie algebras are "three graces", according to Loday. Their exclusive role among all varieties of algebras from the operadic viewpoint.

6.3. **An alternative approach to cohomology of current Lie algebras.** Symmetrization of variables and substitution in the cocycle equation $d(\sum_{i_in I} \varphi_i(x_1, \dots, x_n) \otimes \alpha_i(a_1, \dots a_n)) = 0$. Formula for $H^1(L \otimes A, L \otimes A)$. Cauchy formula, Young diagrams, representation of the Chevalley-Eilenberg complex computing cohomology of $L \otimes A$ in terms of the Young graphs.

6.4. **Short exact sequences of modules.** Interpretation of equivalence classes of short exact sequences of L-modules $0 \to A \to ? \to B \to 0$ as $H^1(L, Hom(B, A))$.

LECTURE 7

7.1. **Cohomology long exact sequence associated to the short exact sequence of modules.** Construction of the Bockstein homomorphism.

7.2. **Whitehead lemmas.** Sketch of possible proofs using induction on dimension of modules using the cohomology long exact sequence, and using induction on dimension of an algebra using $sl(2)$-subalgebras. Converse to Whitehead lemmas.

7.3. **Spectral sequences.** Spectral sequence abutting to cohomology of a filtered complex. Leray, history of invention of spectral sequences.

7.4. **Hochschild–Serre spectral sequence.**

LECTURE 8

8.1. **Künneth formula.** $H^*(L_1 \otimes L_2) \simeq H^*(L_1, K) \otimes H^*(L_2, K)$. Proof via the Hochschild–Serre spectral sequence

8.2. **Tensor product of complexes.** Tensor product of graded vector spaces, Künneth theorem in general.

8.3. **Another example of application of Hochschild–Serre spectral sequence.** Computation of $H^3(sl(2), K)$ taking subalgebra $\langle h \rangle$.

8.4. **Cyclic cohomology.** Direct definition via cyclic subcomplex of the Hochschild complex computing $HH^*(A, A^*)$.

8.5. **Lie algebra homology.** Chevalley–Eilenberg chain complex, duality between homology and cohomology, concrete formulas for $H_1(L, K)$ and $H_2(L, K)$.

LECTURE 9

9.1. **Refresher: nilpotent and solvable Lie algebras.**

9.2. **Refresher: free algebras, generators and relations.**

9.3. **Interpretation of $H_1(L, K)$ and $H_2(L, K)$ for nilpotent L as generators and relations.**

9.4. **$H_2(L, K)$ for perfect L as the universal central extension.**

9.5. **Appearance of Hopf algebras in cohomology.**

Combination of Künneth theorem and pieces of cohomological long exact sequence, for a bialgebra A gives multiplication in cohomology $HH^*(A, A) \otimes HH^*(A, A) \to HH^*(A, A)$.

Hopf algebra structure on $H^*(gl(A), K)$.

Interpretation of non-coboundary Lie coalgebra structures on a Lie algebra L as $H^1(L, L \wedge L)$.

9.6. **Universal enveloping algebra.** Its significance, Poincaré–Birkhoff–Witt theorem, Hopf algebra structure.

LECTURE 10

10.1. **Consequences of Whitehead lemmas.** The first Whitehead lemma implies that every finite-dimensional representation of a finite-dimensional semisimple Lie algebra over a field of characteristic zero is a direct sum of irreducible representations; in particular, any such algebra is a direct sum of simple ones.

The second Whitehead lemma implies Levi–Malcev decomposition.

10.2. **Review of available computer programs.** Mathematica and Maple vs. GAP and Sage. SuperLie. Albert.

10.3. **Some open problems.** Cohomology of the Poisson (= of Hamiltonian vector fields) algebra. Commutative cohomology in characteristic 2. Lie algebras of cohomological dimension 1. Lie superalgebras and their cohomology.

Email address: pasha.zusmanovich@gmail.com