5020 MODERN ALGEBRA

PASHA ZUSMANOVICH

This is, actually, a course on Lie algebras. Each lecture lasted 1 h 15 min.

Lecture 1

Place of Lie algebras in mathematics.
Sophus Lie.
Nonassociative algebra, notion of subalgebra.
Definition of a Lie algebra.
Lie algebra associated with an associative algebra (with proof). Example: $gl_n(K) = M_n(K)^{(-)}$.
Definition of derivation. Theorem: $Der(A)$ is a Lie algebra for arbitrary algebra A (with sketch of the proof).

Lecture 2

$D(1) = 0$ for any derivation D of a unital algebra.
Computation of $Der(K[x])$. One-sided Witt algebra.
Multiplication table.
Abelian Lie algebras, commutant.
Theorem: Every Lie algebra of dimension 2 is either abelian, or isomorphic to a 2-dimensional algebra $\langle x, y \,|\, [x, y] = x\rangle$.

Lecture 3

Properties of the 2-dimensional nonabelian Lie algebra.
Derived series, notion of a solvable Lie algebra.
Two-dimensional subalgebras of the (one-sided) Witt algebra.
Isomorphisms, automorphisms, automorphism group of a given Lie algebra.

Lecture 4

Structure constants of a Lie algebra.
Derivations and automorphisms of the 2-dimensional nonabelian Lie algebra.
$Aut(\text{abelian}) \simeq GL_n(K)$.
Relationship between derivations and automorphisms: $e^D = 1 + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \dots$ (Proof in the case $D^2 = 0$).

Lecture 5

$D^n([x, y]) = \sum_{k=0}^{n} \binom{n}{k} [D^k(x), D^{n-k}(y)]$. Corollary: in characteristic p, D^p is a derivation for any derivation D.
Inner derivations.
Center of a Lie algebra. $Z(L) = 0$ for L two-dimensional nonabelian.
Ideal of a Lie algebra. $Z(L)$ and $[L, L]$ are ideals.

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Lecture 6

All derivations of the 2-dimensional nonabelian algebra are inner.
Fragments of the classification of 3-dimensional Lie algebras (no full proofs).
3-dimensional Heisenberg algebra.
Lower central series, nilpotent Lie algebras.
Direct sum of Lie algebras.
Lemma: A Lie algebra with an ideal of codimension 1 cannot be perfect.
Simple Lie algebras, their importance in the classification theory.
Lemma: A 3-dimensional perfect Lie algebra is simple.

Lecture 7

Quotients by an ideal. Examples involving 2-dimensional nonabelian and 3-dimensional Heisenberg algebras.
Computations of derivations and automorphisms of 3-dimensional Heisenberg.

Lecture 8

Theorem: One-sided Witt algebra is simple.
Starting classification of 3-dimensional simple Lie algebras over an algebraically closed field of charactersitic $\neq 2$.

Lecture 9

Finishing classification of 3-dimensional simple Lie algebras over an algebraically closed field of characteristic $\neq 2$.
Centers of an associative algebra and its associated Lie algebra coincide.
$gl(n)$, $sl(n)$, their relationship.

Lecture 10

Theorem: Any finite-dimnesional Lie algebra over an algebraically closed field either has all adjoint maps nilpotent, of contains a 2-dimensional nonabelian subalgebra.
Realization of $sl(2)$ via traceless matrices.
$sl(2)$ as a subalgebra of the Witt algebra W.
Graded Lie algebras, their subalgebras and associated graded subalgebras.
Theorem: Any finite-dimensional subalgebra of the Witt algebra is either 1-dimensional, or 2-dimensional abelian, or 3-dimensional simple. (Start of the proof).

Lecture 11

Finish of the proof of theorem about finite-dimensional subalgebras of the Witt algebra.
$Aut(sl(2)) \simeq PGL(2)$ (without proof).

Lecture 12

Introduction to GAP.
Inner and outer derivations.

Lecture 13

Semisimple and nilpotent elements, tori, root space decompostion.
Examples: $sl(2)$, $sl(3)$.

Lecture 14

Root systems for $sl(n)$.

Lecture 15

$sl(2)$-triples in classical simple Lie algebras.
Representations of Lie algebras.
First homomorphism theorem.
Faithful representation. Adjoint representation, trivial representation.

Lecture 16

Submodule, quotient module, irreducible module, direct sum of modules.
Theorem. Every 1-dimensional representation of a perfect Lie algebra is trivial.
1-, 2 and 3-dimensional representations of $sl(2)$, their natural realizations.
Representation of $sl(2)$ arising from its embedding into W.

E-mail address: pzusmanovich@nccu.edu