METHODS OF MATHEMATICAL PHYSICS. SYNOPSIS OF GRADUATE COURSE AT TTU, FALL 2011

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Lecture 0

A short 10-15 minutes introduction.

Place of Lie groups and Lie algebras within Mathematics, their relationship with other subjects, their significance.

Lecture 1. A glimpse into Galois theory. I

After [G, Chapter 1]. *Good additional readings are:* [S], [L, Chapter VI, §§1-3], [Mi], *or* [W, Chapter 8].

Main theorem of Galois theory about solubility in radicals. "Galois correspondence" as organizing principle in mathematics:

(i) Classical Galois theory of algebraic equations.
(ii) Lie's theory of differential equations.
(iii) Galois theory of databases, etc.

Groups. Elementary examples: S_n, $GL_n(\mathbb{C})$. Multiplication (Cayley) tables. Normal subgroups, simple, solvable, abelian groups. Linear representations of groups.

Symmetries of algebraic equations. Galois group as automorphism group of a field extension.

Quadratic equations: S_2 is abelian.

Cubic equations: S_3 is solvable. Parity of a permutation. Homomorphism $S_n \to \{-1, 1\}$[1].

Alternating group A_n.

Homework: find an error at page 8[2] of [G].

Lecture 2. A glimpse into Galois theory. II

More examples of Galois groups:

(i) $Gal(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}_2$ (nontrivial automorphism generated by conjugation);
(ii) $Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \simeq \mathbb{Z}_2$;
(iii) Galois group of polynomial $x^n - 1$ is equal to $(\mathbb{Z}_n)^*$.

Elementary symmetric polynomials.

Quartic equation: S_4 is solvable: $S_4 \rhd A_4 \rhd V_4 \rhd \{e\}$.

Quintic and higher equations : A_5 is simple and hence S_5 is not solvable.

Facts.

(i) A_n is simple iff $n = 3$ or $n \geq 5$[3].
(ii) S_n is simple iff $n = 2$.
(iii) S_n is solvable iff $n < 5$.

Date: last modified March 21, 2016.

[1] For the, possibly, simplest and most elegant proof of this homomorphism, see [Ol]. Footnotes usually contain material not presented during lectures and added afterwards.

[2] In the published version! In pdf files at the author's homepage, that corresponds to pages 8–9 in the respective file (Chapter 1).

[3] For a proof (similar to those presented by Alari), see, for example, [Mo].

In the class of finite groups: $\{\text{simple}\} \cap \{\text{abelian}\} = \{\text{simple}\} \cap \{\text{solvable}\} = \{\mathbb{Z}_p, p \text{ prime}\}$.

Direct product of groups.

Algorithm of constructing of formulas for roots, by example of quadratic equations. Characters of a group.

Lecture 3. Lie groups. I

After [G, Chapter 2].

Sophus Lie. Notion of a Lie group.

Manifold, charts, atlas, dimension. Examples of manifolds: $\mathbb{R}^*$, S^3, etc.

$GL(2,\mathbb{R})$, $SL(2,\mathbb{R})$. Normal subgroups in these groups. Difference between notions of simplicity as an abstract group and as a Lie group.

Parametrization of those groups. $SL(2,\mathbb{R})$ as a "glued in two points" product of two-sheeted hyperboloid and a circle[4].

Homework: find incorrectnesses at pages 26–27[5] of [G].

Lecture 4. Lie groups. II

Compactness. $\mathbb{R}$ and S^1 are compact, $SL(2,\mathbb{R})$, $GL(2,\mathbb{R})$ are noncompact.

Commutator. Commutant of a group. Abelian groups $\iff$ groups with trivial commutant.

Example: $[GL(2,\mathbb{R}), GL(2,\mathbb{R})] = SL(2,\mathbb{R})$. $[S_n, S_n] = A_n$[6]. For a simple nonabelian group G (e.g., A_n), $[G,G] = G$.

Lower central series, derived series, nilpotence, solvability (second, equivalent definition in terms of derived series).

$\{\text{abelian}\} \subset \{\text{nilpotent}\} \subset \{\text{solvable}\}$.

Example of a nilpotent group: Heisenberg group $Nil(3) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} | a,b,c \in \mathbb{R} \right\}$. Example of a solvable non-nilpotent group: $UT(2) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,b,c \in \mathbb{R}, a,c \neq 0 \right\}$. They are both noncompact too.

Lecture 5. Lie groups. III

5.1. Examples of nonabelian compact Lie groups.

$O(n,\mathbb{R})$, $SO(n,\mathbb{R})$.

$SO(1,\mathbb{R})$ is trivial, $SO(2,\mathbb{R})$ is homeomorphic to the circle, $SO(3,\mathbb{R})$ is homeomorphic[7] to $\mathbb{R}P^3$. They are compact.

5.2. Lie algebras. *After* [G, §§4.1, 4.2].

Linearization as one of the main mathematical ideas. Lie algebra as a tangent space at the unit of a Lie group.

Linearization of $SL(2,\mathbb{R})$ at the neighborhood of E: matrices of trace 0. They are not closed under matrix multiplication, so taking just a matrix multiplication as an operation in a Lie algebra does not fit.

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[4] For some attractively-looking parametrizations of $SO(3,\mathbb{R})$, see `http://mathoverflow.net/questions/70154/matrix-expression-for-elements-of-so3` .

[5] Pages 27–28 in the pdf file.

[6] In fact, every element of $[S_n, S_n]$ is exactly one commutator, and not just a product of commutators. For a simple proof, see [Or, Theorem 1]. This is true also for many other groups (but not true in general).

[7] For a good informal explanation of this homeomorphism, see Wikipedia: [W-C, Section *The hypersphere of rotations*] and [W-R, Section *Topology*].

Lecture 6. Lie algebras. I

Approximation of multiplicative commutator gives an additive commutator.
Jacobi idenitity. Abstract notion of a Lie algebra.
Commutator of a Lie algebra. Abelian Lie algebras.
Classification of 1- and 2-dimensional Lie algebras.
3-dimensional examples: $sl(2, K)$ and 3-dimensional nilpotent (Heisenberg) Lie algebra.

Lecture 7. Lie algebras. II

Significance of the Heisenberg Lie algebra in quantum mechanics: its commutation relations imply uncertainty principle.

Many questions in structure theory of Lie algebras essentially boil down to linear algebra.

Ideals in Lie algebras. Nilpotency, solvability, simplicity. Examples (2-dimensional nonabelian Lie algebra is solvable, 3-dimensional Heisenberg algebra is nilpotent, $sl(2)$ is simple).

Theorem. For every finite-dimensional Lie algebra L over an algebraically closed field, one of the following holds:

(i) L is abelian;
(ii) L contains 2-dimensional nonabelian subalgebra;
(iii) L contains 3-dimensional Heisenberg subalgebra.

A proof modulo Engel theorem.
Homework: Try to find a proof of this theorem using only elementary linear algebra.

Lecture 8. Lie algebras. III. Exponentiation

8.1. **Lie algebras all whose proper subalgebras are 1-dimensional.** Example: $\mathfrak{so}(3)$. Two extreme cases: lattice of subalgebras is "as small as possible" for such algebras, and "as big as possible" for abelian Lie algebras. Existence of infinite-dimensional Lie algebras all whose proper subalgebras are 1-dimensional is an interesting (and difficult) open problem.

To prove that all such *finite-dimensional* Lie algebras are of dimension ≤ 3 over *arbitrary* field is also on open problem, albeit quite doable one, modulo existing literature.

8.2. **Exponentiation.** *According to* [G, §§4.3, 7.1, 7.2].

Exponentiation: Lie algebras → Lie groups as an oppostite operation to linearization. Motivation: we are trying to "move away" from the identity. Topology is not suitable for that, so we are relying on algebra, multiplying elements in the neighborhood of E of the form $E + \varepsilon X$ "many" times. In the limit we get $exp(X)$.

Properties of $exp(X) = 1 + X + \frac{1}{2!}X^2 + \dots$. Due to the Cayley-Hamilton theorem, this always reduces to the sum of the first $n-1$ powers, accumulating coefficients of infinite series of numbers. Utility of the Jordan normal form for calculating of $exp(X)$.

Example: $sl(2, \mathbb{R})$. Appearence of *cosh* and *sinh*. One exponential map is not enough, two are enough (acting on the linear subspaces spanned by matrices $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ and $\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$, mapping them to the two-sheeted hyperboloid and the circle, respectively).

Lecture 9. Lie algebras and Lie groups. IV

9.1. $\det(e^X) = e^{\mathrm{tr}(X)}$. A proof using Jordan normal form.

9.2. [G, p. 106] $so(3)$ and $su(2)$ – example of two isomorphic (over $\mathbb{R}$) Lie algebras with nonisomorphic Lie groups ($SO(3, \mathbb{R})$ and $SU(2, \mathbb{R})$). **Homework 1:** Prove this.

9.3. Extension of the base field. $su(2)$ and $sl(2)$ are non-isomorphic over $\mathbb{R}$, but isomorphic over $\mathbb{C}$. **Homework 2:** Prove this.

9.4. Faithful representations of a Lie algebra. Ado's theorem.

9.5. [W-B] Baker–Campbell–Hausdorff–Dynkin formula. Example of computation for a two-dimensional nonabelian Lie algebra.

Lecture 10. Lie's theory of symmetries of differential equations

Closely after [G, Chapter 16].

Similarity and dissimilarity between Galois and Lie theories.

The constant C in the solution $y = \int_0^x f(t)dt + C$ of the equation $\frac{dy}{dx} = f(x)$ viewed as 1-parametric Lie group (isomorphic to $\mathbb{R}$) acting on solutions of this equation.

Fact. Every 1-dimensional connected Lie group is isomorphic either to $\mathbb{R}$, or to S^1. [8]

Main steps in Lie's approach taking $x\frac{dy}{dx} + y - xy^2 = 0$ as an example.

Homework. Prove that the corresponding 1-dimensional Lie group is isomorphic to $\mathbb{R}$.

Lecture 11. Homeworks

20-30 minutes.

Solutions or outline of solutions of homeworks. Non-split extensions of groups leads to homological algebra. Questions of isomorphisms between Lie algebras can be reduced to system of quadratic equations which can be solved on computer.

Homework. Do there exist ordinary differential equations of the first order whose Lie group of symmetries is isomorphic to S^1?

References

[C] P.M. Cohn, *Lie Groups*, Cambridge Univ. Press, 1957 (reprinted in 1965).

[G] R. Gilmore, *Lie Groups, Physics, and Geometry*, Cambridge Univ. Press, 2008.

[L] S. Lang, *Algebra*, 3rd ed., Springer, 2002 (there exists a Russian translation from the 2nd ed.).

[Mi] J.S. Milne, *Fields and Galois Theory*, Version 4.22, 2011; `http://jmilne.org/math/CourseNotes/ft.html` .

[Mo] P.J. Morandi, *A_n is simple*; `http://sierra.nmsu.edu/morandi/notes/an-is-simple.pdf` .

[Ol] R.K. Oliver, *On the parity of a permutation*, Amer. Math. Monthly **118** (2011), 734–735.

[Or] O. Ore, *Some remarks on commutators*, Proc. Amer. Math. Soc. **2** (1951), 307–314.

[S] I. Stewart, *Galois Theory*, 3rd ed., Chapman & Hall/CRC, 2004.

[W] B.L. van der Waerden, *Algebra*, Vol. I, Frederick Ungar Publ. Co., 1970 (translated from the German 7th ed., 1966; republished by Springer, 2003; there exists a Russian translation).

Wikipedia articles

[W-B] *Baker–Campbell–Hausdorff formula.*

[W-C] *Charts on* $SO(3)$.

[W-R] *Rotation group.*

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[8] The proof can be found in [C, Chapter 2, §2.9].