On the last question of Stefan Banach

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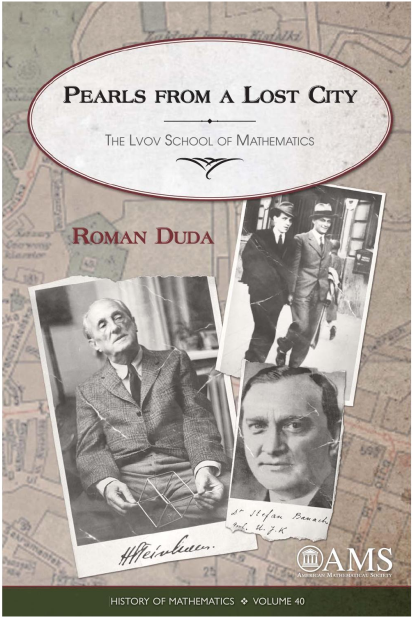
The last question of Banach

Andrzej Alexiewicz's diary, December 29, 1944:

> "There exists a nontrivial example of ternary multiplication, which is not generated by binary multiplication (Banach). Can any finite set with ternary commutative multiplication be extended so that ternary multiplication is generated by a binary multiplication?"

"A lost city"


Pearls from a Lost City
The Lvov School of Mathematics
Roman Duda
Dr Stefan Banach
AMS
American Mathematical Society
HISTORY OF MATHEMATICS ❖ VOLUME 40

Banach's main occupation at that time

Lice feeders at the laboratory of Rudolf Weigl, Lvov, circa 1942-1944

First interpretation of Banach's question

"Multiplication": an arbitrary map: $f : X \times X \times X \to X$

"Commutativity":

$$f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = f(x_1, x_2, x_3)$$

for any $\sigma \in S_3$.

"Generation": $f(x, y, z) = (x * y) * z$ or $x * (y * z)$

"Verification" of Banach's claim

An elementary counting:

$$2|X|^{|X|^2} < |X|^{|X|^3}.$$

Example 1

$a, b \in X$, $a \neq b$

$$f(x,y,z) = \begin{cases} a, & \text{if all } x, y, z \text{ are distinct from } a \\ b, & \text{if at least one of } x, y, z \text{ coincides with } a. \end{cases}$$

Example 2

$X = GF(q)$. All multiary maps on $GF(q)$ are polynomials of degree $< q$ in each variable. Given a polynomial $f(x,y,z)$, to find a polynomial $g(x,y)$ such that $f(x,y,z) = g(g(x,y),z)$, amounts to solving a system with q^3 equations in q^2 unknowns.

Not so elementary counting?

$\lvert X\rvert = n$	number of binary maps $= n^{n^2}$	number of ternary maps $= n^{n^3}$	number of ternary maps $(x * y) * z$	number of ternary maps $(x * y) * z$ and $x * (y * z)$	number of commutative ternary maps $(x * y) * z$	number of commutative ternary maps $(x * y) * z$ with commutative $*$
1	1	1	1	1	1	1
2	16	256	14	21	5	5
3	19683	7625597484987	19292	38472	48	48
4	4294967296	$\approx 3.4 \times 10^{38}$	?	?	?	?

$\stackrel{?}{=}$

Submitted to OEIS by N.J.A. Sloane!

An ultimate answer to Banach's question?

Theorem (Zusmanovich 2016)

For any ternary map $f : X \times X \times X \to X$ there is $Y \supset X$ and a binary map $* : Y \times Y \to Y$ such that $f(x, y, z) = (x * y) * z$.

Proof

Set $Y = X \cup (X \times X)$ and

$$x * y = (x, y)$$
$$(x, y) * z = f(x, y, z).$$

Second interpretation of Banach's question

"Multiplication": ternary semigroup, i.e.

$$f : X \times X \times X \to X$$
$$f(f(x,y,z),u,v) = f(x,f(y,z,u),v) = f(x,y,f(z,u,v)).$$

Theorem (Łoś 1955, Monk–Sioson 1966)

Every (commutative) ternary semigroup can be extended to a (commutative) ternary semigroup given by

$$f(x,y,z) = (x * y) * z,$$

where $*$ is a (commutative) binary semigroup.

Jerzy Łoś (student at Lvov, 1937–1939)

Second interpretation of Banach's question

Idea of the proof
(Zusmanovich 2016, based on an idea of Jacobson 1949)

Consider

$$X \cup \{\text{the semigroup generated by maps } m_{x,y} : z \mapsto f(x,y,z)\}.$$

Multiplication:

$$\begin{aligned} x * y &= m_{x,y} \\ x * g &= g * x = g(x) \\ g * h &= g \circ h. \end{aligned}$$

Extension of Banach's question

What if instead of $(x * y) * z$ we take $(x * y) \circ z$ for two operations $*$ and $\circ$?

or

"Are there actually functions of 3 variables?"

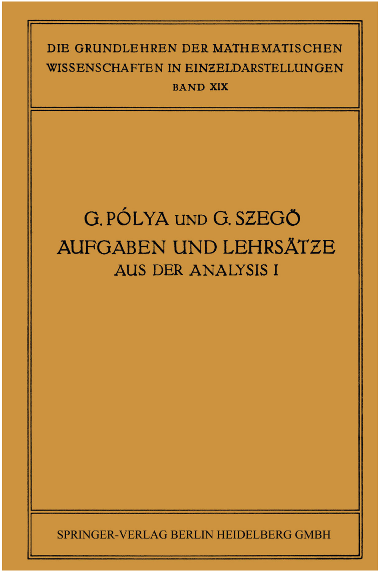
Theorem (Pólya and Szegö 1925, Sierpiński 1945)

For any binary injection $g : X \times X \to X$, and any ternary map $f : X \times X \times X \to X$, there is $h : X \times X \to X$ such that $f(x, y, z) = h(g(x, y), z)$.

Proof

If $u \notin \operatorname{Im} g$, set $h(u, z)$ for any z arbitrarily. If $u = g(x, y)$ for (unique) x, y, set $h(u, z) = f(x, y, z)$.

A timeless classic



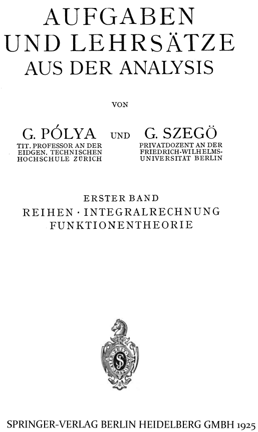
DIE GRUNDLEHREN DER MATHEMATISCHEN WISSENSCHAFTEN IN EINZELDARSTELLUNGEN

BAND XIX

G. PÓLYA UND G. SZEGÖ

AUFGABEN UND LEHRSÄTZE AUS DER ANALYSIS I

SPRINGER-VERLAG BERLIN HEIDELBERG GMBH



AUFGABEN UND LEHRSÄTZE AUS DER ANALYSIS

VON

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TIT. PROFESSOR AN DER EIDGEN. TECHNISCHEN HOCHSCHULE ZÜRICH

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ERSTER BAND

REIHEN · INTEGRALRECHNUNG FUNKTIONENTHEORIE

SPRINGER-VERLAG BERLIN HEIDELBERG GMBH 1925

Wacław Sierpiński (Professor at Lvov, 1908–1914)

Hilbert's 13th problem

A curious fact

In the class of continuous functions, the theorem is not true.

Hilbert's 13th problem

Does there exist a real continuous function in 3 variables which cannot be represented as a superposition of real continuous functions in 2 variables?

Another curious fact

In the class of smooth functions, if $F(x, y, z) = f(g(h(x, y), z), z)$, then

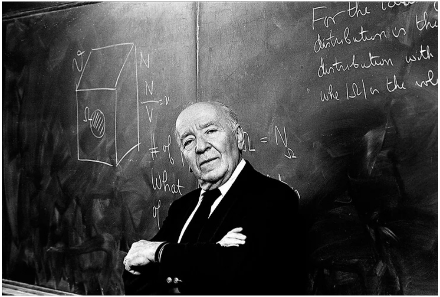
$$\frac{\partial^2 F}{\partial x \partial z}\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial y \partial z} = 0.$$

Answer to Hilbert's 13th problem (Kolmogorov, Arnold, 1956-1959)

No.

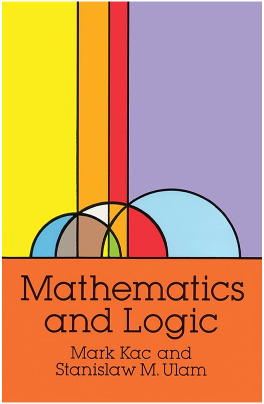
Lvov connections to Hilbert's 13 problem?

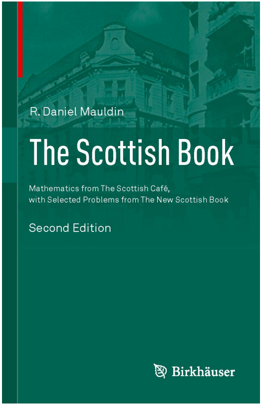
Stanisław Ulam
(PhD Lvov 1933)

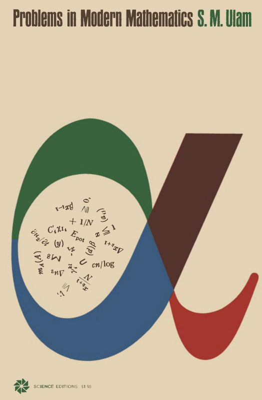


Mark Kac
(PhD Lvov 1937)

Some more literature







Ulam: "... the pleasure of past collaboration with Banach ..."

Kac: "I was less of a habitue of the Scottish Café... I was financially somewhat less affluent than Stan – I was ... independently poor. And it did cost a little to visit in the Café"

~~Second~~Third interpretation: clones

"Multiplication": an arbitrary map $X \times X \times X \to X$
"Generation": superposition (like in Hilbert's 13th)

Theorem (Sierpiński 1935, Banach 1935)

Any countable number of unary maps on an infinite set can be generated by two maps.
(or: A countable transformation semigroup is 2-generated).

Generalizations:

- Other kinds of semigroups, generation "up to approximation" (Schreier–Ulam, Jarník–Knichal, et al.)
- Unary $\to$ multiary

Józef Schreier (PhD Lvov 1934)

~~Second~~Third interpretation: clones

Theorem

Any countable number of maps of arbitrary arity on a set X can be generated by one binary map on X.
(or:
The clone of all maps on X is generated by its binary fragment).

Webb 1935: finite X (generalization of Sheffer's stroke)

$$p \uparrow q = \neg(p \wedge q)$$

Łoś 1950, Goldstern 2012: infinite X

Conclusion

Bad interpretation.

~~Second~~Third interpretation: clones

Idea of the proof for infinite X

1. Could be reduced to binary functions by composing with iterations of the (generalization of) Sheffer stroke. Let $\{g_1, g_2, \dots\}$ be the set of binary functions under the question.
2. Set $X = Z \times \mathbb{N}$, choose a bijection $i : X \to Z \times \{0\}$, and assuming $x = (z, n) \in X$, write $x + k$ for $(z, n + k)$. Find $f : X \times X \to X$ such that:

(i) $f(x, x) = x + 1$

(ii) $f(x, x + 1) = i(x)$

(iii) $f(i(x) + k, i(y)) = g_k(x, y)$.

Erdös (on another occasion, from a preface to the *Scottish Book*):

> "Now it frequently happens in problems of this sort that the infinite dimensional case is easier to settle than the finite dimensional analogues. This moved Ulam and me to paraphrase a well known maxim of the American armed forces in WWII: 'The difficult we do immediately, the impossible takes a little longer', viz: 'The infinite we do immediately, the finite takes a little longer'."

Another Polish speciality of that time

Łukasiewicz' 3-valued logic $Ł$:

	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

E.T. Bell (1934):

> "Looking back over the 6000 years we have come since the beginning of definitely recorded history, we see four great peaks towering above the general level of profound or lofty speculation on the nature of truth. ... the fourth, rising in 1930, the creation, by Lucasiewicz and Tarski, of numerous patterns of strict deductive reasoning radically different from the traditional logic of Aristotle."

The major question: Functional completeness.

Another Polish speciality of that time

Theorem (Słupecki 1936)

Let $X = \{0, \frac{1}{2}, 1\}$, N is the "negation" $0 \mapsto 1, \frac{1}{2} \mapsto \frac{1}{2}, 1 \mapsto 0$, and T is the unary map sending everything to $\frac{1}{2}$.

(i) There are multiary maps $X \times \cdots \times X \to X$ which cannot be represented as a superposition of $Ł$ and N.

(ii) Any multiary map $X \times \cdots \times X \to X$ is a superposition of $Ł$, N, and T.

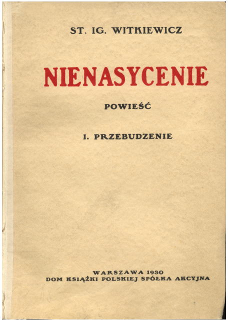
Ernst Nagel (1936):

> "Professor Lukasiewicz's seminar at Warsaw was crowded with competent young men, incomparably better equipped in logic than students of like age in America, who were expected to write as seminar exercises papers which elsewhere would be thought important enough for publication."

And, finally, more pictures, literature, and quotes

Jan Łukasiewicz

Stanisław Ignacy Witkiewicz



"Benz ... w krótkim czasie doszedł do zadziwiających wyników: oto z jednego jedynego aksjomatu, którego nikt prócz niego nie rozumiał, zbudowal całkiem nową logikę i w jej terminach określał całą matematykę, sprowadzając wszystkie definicje do kombinacji paru znaczków podstawowych."

The End