Seminar 1. Some geometric applications of Hodge theory

Thursday Nov. 6, 2020. @ 17:00.

Speaker: Phillip Griffiths.

I. Introduction
II. Origins of Hodge Theory.
III. Objects of Hodge
IV. Some geometric applications of Hodge Theory

* Introduction:

Modern Hodge theory is both a subject of study in its own right and a subject that is used in many areas of current mathematical research, especially in but no means restricted to algebriac geometry. The talk is a practical and informal overview of some of its uses with emphasis on those in algebriac geometry. The talk discusses some of the historical development of the subject, how did it originate and how did it get to the current state? The emphasis will be on the period up until the time of Hodge and only touches on a few of the major milestones.

Given in the Appendix: a proof of the local Torelli theorem for I. surfaces (the "first" non classical general type surface). This to illustrate one computational technique in Hodge Theory.

Hodge Theory relates to many areas in mathematics

$\Rightarrow$ Analysis
- classical complex analysis
- modern complex analysis (the $\bar{\partial}$ operator)
- ODE's (Picard-Fushs equations)
- non-linear PDE's (non-abelian Hodge theory and special metrics on algebriac varieties.

$\Rightarrow$ Differential Geometry: provides a tool for the uses of Hodge theory in algebriac geometry including vanishing theorems, general type properties of algebriac varieties including moduli spaces, hyperbolicity of practical varieties.

and finally of course

$\Rightarrow$ Algebriac Geometry

very few aspects of algebriac geometry discussed, the topics did not invalve the central problem in the subject (Hodge conjucture) mirror symmetry and other topics related to physics, character varieties, arithematic algebriac geometry, ...

$\Rightarrow$ Specifically, the use of Hodge theory to study moduli requires using geometric constructions arising from Hodge theory, frequently some type of Torelli property, either for the variety itself or for the singular ones that appear on the boundary of moduli spaces. A Hodge structure and some of its generalizations are given by linear algebra data. In some cases, from the linear algebra data one may construct a geometric object from which properties of the variety may be determined. Although very classical this original use of Hodge theory (elliptic functions and Riemann's theta function) continues to find applications to geometric questions including moduli.

II. Origins of Hodge Theory.

Although there is no single work that one can say "here is the beginning of "Hodge theory," an important part of its origins may be traced to the study of integrals

(II.1) $$\int r(x, y(x))\, dx$$

of algebriac function. Here $y(x)$ is the function given by an equation

$$f(x, y(x)) = 0$$

where $f(x,y)$ is an irreducible polynomial and $r(x,y) = p(x,y)/q(x,y)$ is a rational function. Such integrals arise in geometry, e.g. as arclength.

$$\int \sqrt{dx^2 + dy^2} = \int \sqrt{(1+(y'(x)^2)}\, dx$$ or in mechanics

where $y(t^2) = f(y(t))$ so that $y(t) = \sqrt{f(y(x))}\, dx$

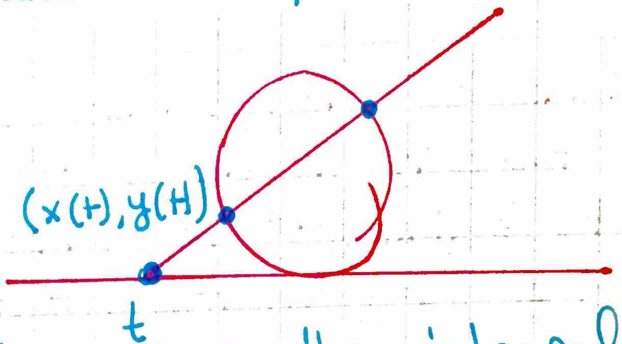
for illustrative purposes we will take

(II.2) $$f(x,y) = y^2 - P(x)$$

where $P(x) = \prod_{i=1}^{2g+2}(x - a_i)$ is a polynomial distinct roots. The integral (II.1) is understood to take place in the complex plane along a path γ avoiding the a_i and along which we have chosen a branch of $y(x) = \sqrt{P(x)}$. If $\deg f(x,y) = 2$, then such integrals can be evaluated in terms of "elementary functions" for example, for $f(x,y) = x^2 + y^2 - 1$ then,

(11.3) $$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dy}{y} = \arcsin.$$

The reason for $\deg f = 2$ is that $f(x,y)=0$ is then a conic and can be parametrized by a line.



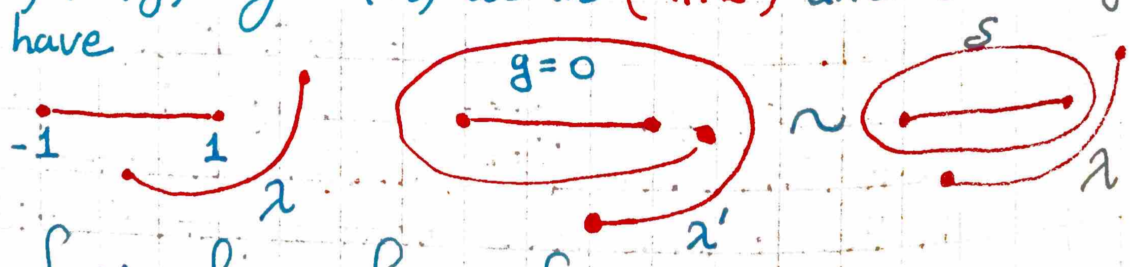
substituting gives the integral $\int r(t)\,dt$ where $r(t)$ is a rational function and partial functions may be used to evaluate it.

As soon as $\deg f \geqq 3$ the integral (11.1) can no longer be understood in terms of elementary functions. for example:

(11.4) $$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad k \neq \pm 1$$

turns up in the arc length of an ellipse.

* Observation that has both topological and analytic meaning.
For $f(x,y) = y^2 - P(x)$ as in (11.2) and $\omega = dx/y$ we have



$$\int_{\lambda'} \omega = \int_{\lambda} \omega + \int_{\delta} \omega = \int_{\lambda} \omega + c$$

where $c = \int \omega$ is a period of the integral.

If we invert the integral by setting

(11.5) $$u = \int_{(x_0, y_0)}^{(x(u), y(u))} \omega .$$

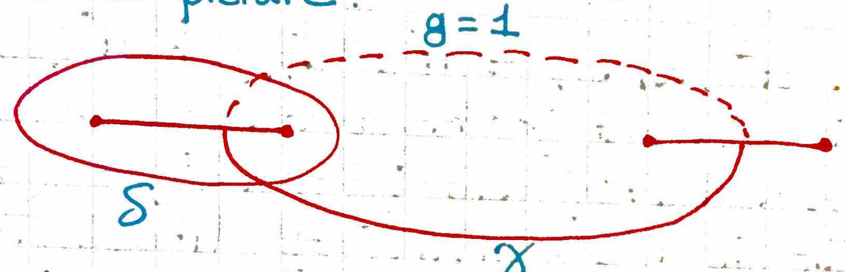
then $x(u+c) = x(u)$, $y(u+c) = y(u)$

In fact, in (11.3) we have $c = 2\pi$ and this shows that defining $x(y)$, $y(u)$ by (11.5) parametrizes the circle by arclength and gives periodicity of $(\sin u, \cos u)$.

Note that taking the derivative of (11.5) gives:

$$x'(u) = y(u)$$

If we try to do the same thing with (11.4), then we arrive at a picture.



where inversion of the integral leads to doubly periodic functions $x(u)$, $y(u)$.

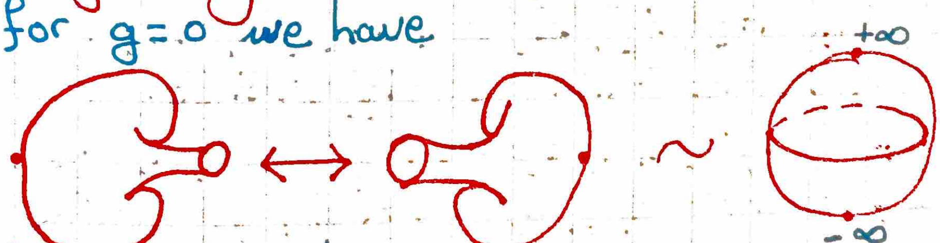
Analytically:

→ for $g=0$ the integral $\int_\gamma \omega \longrightarrow \pm\infty$ as $\partial\gamma \longrightarrow \infty$ (in fact, $\omega \sim \pm(dt/t^{1/2})$ for $x = 1/t$);

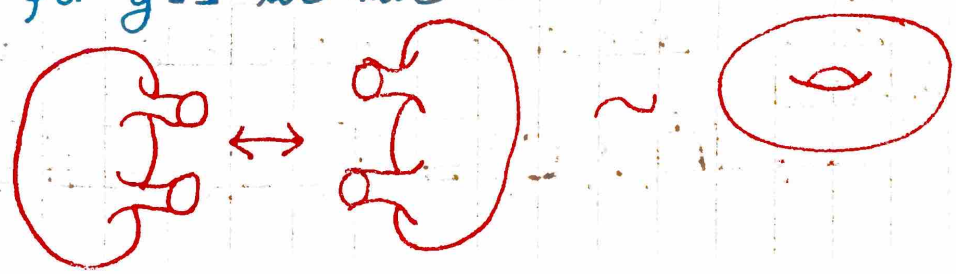
→ for $g=1$ the integral $\int_\gamma \omega < \infty$ for any γ (in fact, for $k \geqq g-1$ we have $\int_\gamma \frac{x^k dx}{y} < \infty$)

Topologically:

→ for $g=0$ we have



→ for $g=1$ we have



If $r(t)\,dt$ has no residues so that $\int r(t)\,dt$ is a rational function, then:

$$x_i(u+c) = R_i(x_1(u), x_1'(u), \ldots, x_d(u), x_d'(u))$$

Taking $h(x,y,t)$ to be linear as above one may obtain the addition theorem

$$x(u+\tilde{u}) = R(x(u), x'(u), x(\tilde{u}), x'(\tilde{u}))$$

→ Abel ([A]) defined the genus of $X := \{f(x,y)=0\}$ as the dimension of the space $H^0(\Omega^1_X)$ of ω's with $\int \omega < \infty$. The Riemann ([R]) proved

→ $g = (\frac{1}{2})\, b_1(X) = \frac{1}{2}(\dim H_1(X,\mathbb{Z}))$

→ $H^1(X,\mathbb{C}) \cong H^0(\Omega^1_X) \oplus \overline{H^0(\Omega^1_X)}$

His argument for the second used $\int \omega \wedge \omega = 0$ (because $dx \wedge dx = 0$) and $i\int \omega \wedge \bar{\omega} > 0$ (because $i\,dx \wedge d\bar{x} > 0$). This is the beginning of Hodge Theory.

→ Finally a few words about Picard. He studied algebriac surfaces.

$$X = \{f(x,y,z)=0\}$$

by looking at the pencil of algebriac curves

$$Y_z = \{f(x,y,z)=0,\ z = \text{constant}\}$$

Then he proved.

→ Y_z is connected $(H_0(Y_z,\mathbb{Z}) \xrightarrow{\sim} H_0(X,\mathbb{Z}))$

→ $H_1(Y_z,\mathbb{Z}) \longrightarrow H_1(X,\mathbb{Z})$.

On X he considered differentials.

→ $\varphi = p(x,y,z)\,dx + q(x,y,z)\,dy$, $\int \varphi < \infty$

→ $\omega = r(x,y,z)\,dx \wedge dy$, $\iint \omega < \infty$

For the first Picard showed that $d\varphi = 0$ and forth this inferred

→ $H^1(Y,\mathbb{C}) \cong H^0(\Omega^1_X) \oplus \overline{H^0(\Omega^1_X)}$

III Objects of Hodge Theory.

→ Polarized Hodge Structure (PHS): (V, Q, F)

→ $V = \mathbb{Q}$- vector space

→ $Q : V \otimes V \longrightarrow \mathbb{Q}$

→ $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$

where

$$F^p \cap \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}, \quad 0 \leqq p \leqq n.$$

for $V^{p,q} = F^p \cap \overline{F}^q$ we have the Hodge decomposition

→ $V_{\mathbb{C}} = \bigoplus^{p+q=n} V^{p,q}$, $\overline{V}^{p,q} = V^{q,p}$

→ $F^p = \bigoplus^{p' \geq p} V^{p',q}$

This defines a Hodge structure of weight n. The polarization arises from the first and second Hodge-Riemann bilinear relations, which are generalizations of the ones given above for algebriac curves and which we don't need to make explicit. Most of the deeper results in Hodge theory require the existence of polarization.

Example: For X a smooth algebriac variety, $H^n(X, \mathbb{Q})$ has an Hodge Structure of weight n. If $X \subset \mathbb{P}^N$, so that $L = \mathcal{O}_X(1)$ is an ample line bundle, there is a polarization on $H^n(X, \mathbb{Q})$.

→ Mixed Hodge structure (MHS): (V, W, F)

→ $V = \mathbb{Q}$ vector space

→ $W_0 \subset \cdots \subset W_m = V$

→ $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$

where F induces a weight k Hodge structure on

$$Gr_k^W(V) = W_k / W_{k-1} := H^k$$

In practice we will have a lattice $V_{\mathbb{Z}} \subset V$ with induced lattices on the H^k's. Thus a MHS is a successive extension of Hodge structures with the first level being a direct sum of terms

$$\mathrm{Ext}^1_{MHS}(H^k, H^{k-1}) = \frac{\mathrm{Hom}(H^k, H^{k-1})}{F^0 \mathrm{Hom}(H^k, H^{k-1}) + \mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-1})}$$

In general the set $\mathcal{E}_k$ of at most k-fold iterated extensions of HS's with fixed H^k's gives a sequence of fibrations $\mathcal{E}_k \longrightarrow \mathcal{E}_{k-1}$ with typical fibre the direct sum of

$$\mathrm{Ext}^1_{MHS}(H^\ell, H^{\ell-k})\text{'s}$$

→ Variation of Hodge Structure (VHS): $(\mathbb{V}, \mathcal{F}, \nabla; B)$

→ $\mathbb{V} = \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$ is a local system over a quasi-projective variety B.

→ $\mathcal{F}$ is a filtration $\mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \cdots \mathcal{F}^0 = V = \mathbb{V} \otimes \mathcal{O}_B$ by sub-bundles that induces a Hodge structure on each fibre.

→ $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_B$ satisfies $\nabla F = 0$ and the infinitesimal period relation (IPR)

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_B$$

It is understood that there is a horizontal bilinear form $Q : \mathbb{V} \otimes \mathbb{V} \longrightarrow \mathbb{Q}$ that polarizes the Hodge structure on each fibre.

Example: $\mathcal{X} \xrightarrow{\pi} B$ is a smooth family of projective algebriac varieties X_b and

→ $\mathbb{V} = R^n_\pi \mathbb{Q}_{\mathcal{X}}$

→ $\mathcal{F}_b$ gives the Hodge structure on $H^n(X_b, \mathbb{C})$

→ ∇ is the Gauss - Manin connection; $\nabla^2 = 0$

→ Infinitesimal variation of Hodge structure **IVHS**

(IVHS): (E, T, θ)

→ $E = \bigoplus E^p$

→ $\theta : E^p \longrightarrow E^{p-1} \otimes T^*$

→ $\theta \wedge \theta = 0$

Thus E is a Sym T- module.

Example: $E^p = Gr^p_{F_b}(V_b)$, $T = T_b B$ and θ is induced by ∇. It is essentially given by the differential of the period mapping (which we won't define here).

→ Limiting mixed Hodge structure (**LMHS**)

(**LMHS**): $(V, W(N), F)$

→ $N \in \mathrm{End}(V)$ satisfying $N^{m+1} = 0$ gives a unique weight filtration $W_k(N)$ satisfying

$$N : W_k(N) \longrightarrow W_{k-2}(N)$$

$$N^k : Gr^{W(N)}_{m+k}(V) \cong Gr^{W(N)}_{m-k}(V)$$

$$N : F^p \longrightarrow F^{p-1}$$

Again we assume there exists a $Q : V \times V \longrightarrow \mathbb{Q}$ and $N \in \mathrm{End}_{\mathbb{Q}}(V)$. Then there are induced bilinear forms

$$Q_k : Gr^{W(N)}_k(V) \otimes Gr^{W(N)}_k(V) \longrightarrow \mathbb{Q}$$

and these are assumed to polarize the Hodge structures on the $Gr^{W(N)}_k(V)$

Example: for $\Delta^* = \{0 < |t| < 1\}$ suppose we have a VHS $(\mathbb{V}, \mathcal{F}, \nabla; \Delta^*)$ over the punctured disc, For $V = \mathbb{V}_{t_0}$ it is known that the monodromy

$$T: V \longrightarrow V$$

is quasi-unipotent. i.e., $T = T_s T_u$ where $T_s^{\ell} = I$ and $T_u = e^N$ with $N^{m+1} = 0$ for som $m \leq n$. Then Schmid proved that there is a LMHS H^n_{lim} (actually an equivalence class of such parametrized by $T^*_{\{0\}}\Delta$).

This example has been extended by Cattani-Kaplan-Schmid to the case when the parameter space $\Delta^{*a} \times \Delta^b := B$. For our purposes it will be convenient to assume the local monodromies around the generators of $\pi_1(\Delta^{*a})$ are unipotent. In this case there is a canonical Delinge extension of the Hodge bundles to $\mathcal{F}^p_e \longrightarrow \Delta^a \times \Delta^b := \overline{B}$ where the Gauss-Manin connection satisfies

$$\nabla: \tilde{\mathcal{F}}^p_e \longrightarrow \tilde{\mathcal{F}}^{p-1}_e \otimes \Omega^1_B(\log Z)$$

Note that the left-hand side is independant of the particular singular fibre X_0 over the origin.

$\rightarrow$ In the global situation where one has a pair $(\overline{B}, Z)$ consisting of a smooth projective variety $\overline{B}$ with $Z \subset \overline{B}$ a reduced normal crossing divisor, given a VHS over B with unipotent monodromies around the irreducible branches of Z the above local discussion applies to give a canonical extension of the VHS to $\overline{B}$.

* Again the assumption of unipotent monodromies is convenient for expository purposes; in interesting geometric situations it frequently does not occur (e.g. algebriac surfaces acquiring normal (and hence isolated singularities)

IV. Some geometric applications of Hodge Theory.

→ In some situations one may associate to a PHS, an IVHS or a LMHS a geometric construction and when the Hodge theoretic object arises from geometry this construction may help understand the geometry:

The classical example: Associated to a weight $n=1$ PHS (V, Q, F) is

→ a compact complex torus
$$J = (F^1 \backslash V_{\mathbb{C}})/V_{\mathbb{Z}};$$

→ a Line bundle $L \longrightarrow J$ with
$$c_1(L) = Q \in (\Lambda^2 V_{\mathbb{Z}})^* \cong H^2(J, \mathbb{Z});$$

→ it follows from the Hodge-Reimann bilinear relations L is holomorphic and positive in the differential geometric sense, hence it is ample. If Q is unimodular, then $h^0(J, L) = 1$ and there is a non-zero section θ giving a divisor $\Theta \subset J$.

This is the case when the PHS arises from the H^1 of a smooth algebriac curve X. There is an Abel-Jacobi map
$$u: X \longrightarrow J(X)$$
given for $p \in X$ by the linear function on $H^0(\Omega^1_X) \cong (F^1/V_{\mathbb{C}})^*$:
$$\omega \longrightarrow \int_{p_0}^{p} \omega \quad \text{modulo periods.}$$

Using Abelian sums as were encountered in Abel's theorem this mapping extends to

$$\begin{array}{ccc} X^{(d)} & \longrightarrow & J(X) \\ \cup & & \cup \\ \Sigma_i\, p_i & \longrightarrow & \Sigma_i\, u(p_i) = \Sigma_i \int_{p_0}^{p_i} \omega \end{array}$$

Riemann [R] proved that, up to a translation,

$$u(X^{(g-1)}) = \Theta.$$

From this one may determine much of the geometry of X culminating in Torelli's theorem that the PHS on $H^1(X)$ determines X.

In general it is not possible to associate a geometric object to a PHS. The reason is the differential constraint imposed on a VHS by the IPR. Only when the period domain is Hermitian symmetric, in which case the IPR is trivial and the PHS is a (Taste twist of) one weight $n=1$ or of weight $n=2$ with Hodge number $h^{2,0}=1$, can we construct naturally an algebro-geometric object from just a PHS. In the higher weight case one frequently may use an IVHS as a surrogate for the Θ-divisor. A recent case of this is given by the

Example (Shepherd-Barron):
If $X \longrightarrow C$ is an elliptic surface with no multiple fibres and with $h^{2,0} \geq h^{1,0} + 3$,
then the PHS and $H^2(X)$ generically determines X. The IVHS used here is the first variation of the PHS on $H^2(X)$, which in a very interesting way is interepreted via the ramification of the j-function map $C \longrightarrow \overline{M}_1$

A classical example of the use of IVHS is the theorem of Donagi-Green ([GMV]) concerning smooth hypersurfaces

$$\{F(x_0, x_1, \ldots, x_{n+1}) = 0\} = X \subset \mathbb{P}^{n+1}$$

In this case, for $n \geq 2$ the IVHS is given by a homogeneous subring $R \subset \mathbb{C}(x_0, ---, x_{n+1})/J_F$ where $J_F = \{F_{x_0}, ---, F_{x_{n+1}}\}$ is the Jacobian ideal of F. Except for a few cases of $\deg F$, it is shown that R determines J_F up to the action of PGL_{n+1}.

As even more classical examples arises in the case of algebriac curves X of genus $g \geq 2$. In this case the IVHS turns out to be equivalent to the map.

$$\mathrm{Sym}^2 H^0(K_X) \longrightarrow H^0(2K_X)$$

for $g \geq 3$ and X non-hyperelliptic this mapping is surjective (Noether) (which then implies local Torelli) with kernel the space $I_2(X)$ of quadrics through the canonical curve $X \subset \mathbb{P}^{g-1}$. For $g \geq 5$, again by Noether

$$\bigcap_{Q \in I_2(X)} Q = X$$

and from this one infers generic global Torelli for $g \geq 5$

* In these examples involving full use of the two Hodge-Riemann bilinear relation is not made. In the following both the IVHS and the bilinear functions are used.

Example: A recent example that is in some ways reminiscent of the Θ-divisor arises from a pair $(\overline{B}, Z)$ together with a VHS over $B = \overline{B}/Z$ that has been canonically extended to $\overline{B}$. We have noted that at each point of Z the level 1 extension data $\mathcal{E}_1$ is a direct sum of compact, complex tori each of whose tangent spaces is a Hodge structure of weight -1 with Hodge decomposition.

$$\text{III.1} \quad (K-1, -K) \oplus --- \oplus \underbrace{(0,-1) \oplus (-1,0)}_{} \oplus --- \oplus (-K, K-2)$$

This gives the picture of the VHS at infinity

$$\begin{tikzcd} \left\{\begin{array}{l}\text{subvarieties of}\\ \text{abelian varieties}\end{array}\right\} \arrow[r] & (\mathcal{V}_e, \tilde{F}_e) \arrow[d, "\left\{\begin{array}{l}\text{associated graded}\\ \text{to the LMHS}\end{array}\right\}"] \\ & \left\{\begin{array}{c}\text{variations of}\\ \text{PHS's}\end{array}\right\} \end{tikzcd}$$

where using the above result in the box the co-normal bunch to the fibres in $\overline{B}$ is expressed by Θ-line bundles along the fibres. We believe that the above picture serves as the governing property in the global understanding of the Hodge structure on the boundary of the canonically extended VHS ([GGR])

Geometric example: Related to the preceding. Let M be a KSBA moduli space of surfaces of general type and with canonical distribution $\overline{M}$ where the boundary $\partial M = \overline{M}/M$ parametrizes surfaces X with semi-log-canonical singularities. A part N_e of the boundary corresponds to normal Gorenstein X's with simple elliptic singularities. In contrast to the case of algebriac curves where $\overline{M}_g$ is essentially smooth, $N_e \subset \partial M$ is generally highly singular and consideration of the extension data in the LMHS suggests a natural partial desingularization of $\overline{M}$ along N_e. This is explained in the notes $[G_2]$, where it is also illustrated that this phenomenon may also extend to non-Gorenstein isolated singularities and to non-normal X's as well.

This is the first non-classical example of where an actual geometric object may be constructed from a (generalized) Hodge structure alone, in this case a LMHS. It is an IVHS because from the work of Cattani-Kaplan-Schmid a LMHS may be smoothed to 1-parameter of ordinary PHS's.

Geometric example (continued): As above suppose we have a KSBA moduli space $\mathcal{M}$ for general type surfaces and with canonical completion $\overline{\mathcal{M}}$. Assume for simplicity that $\mathcal{M}$ is smooth and that a general point of $\mathcal{M}$ corresponds to a smooth regular surface. One may ask

→ can Hodge theory suggest what surfaces X appear on the boundary $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$?

→ can Hodge theory suggest how one might construct a disingularization $\widetilde{\mathcal{M}}$ of $\overline{\mathcal{M}}$?

As discussed and illustrated in [G2] the answer to both questions is positive. For the first, a general limiting mixed Hodge structure has

$$N^2 = 0, \quad \operatorname{rank} N = 2$$

Thus the LHMS has associated graded $(H^1, H^2, H^1(-1))$ where $H^1 = H^1(C)$ for a smooth elliptic curve C. Given a KSBA degeneration $\mathcal{X} \longrightarrow \Delta$ where X_t is smooth for $t \neq 0$ and X_0 is a normal surface corresponding to a point of $\partial\mathcal{M}$, what is suggested that $X_0 = (X, p)$ where p is a simple elliptic singularity of a surface X and where the resolution of that singularity is $(\widetilde{X}, C) \longrightarrow (X, p)$ where $\widetilde{X}$ is smooth and $C \subset \widetilde{X}$ is an elliptic curve.

For the desingularization of $\overline{\mathcal{M}}$ one needs to do a semi stable reduction

$$\widetilde{\mathcal{X}} \longrightarrow \widetilde{\Delta}$$

of the family $\mathcal{X} \longrightarrow \Delta$. Since $N^2=0$, Clemens-Schmid suggests that the central fibre $\mathcal{X}_0$ should have a double curve C and no triple points. The simplest possibility is that

$$\tilde{\mathcal{X}}_0 = \tilde{X} \sqcup_C Y$$

where $\tilde{X}$ is as above and Y is a smooth surface containing C. Since C is a smooth elliptic curve we might try a smooth cubic $C \subset \mathbb{P}^2$. The normal bundle $N_{C/X}$ has degree $d = -C^2$ while $N_{C/\mathbb{P}^2} \cong \mathcal{O}_C(3)$. To achieve the neccessary condition.

$$N_{C/\tilde{X}} \cong \check{N}_{C/Y}$$

for smoothability, we must blow $9-d$ points p_i on C. Since for a smoothable elliptic singularity we have

$$1 \leqq d \leqq 9$$

Y is a del Pezzo surface. Moreover from $\tilde{X}_0$ as above we can construct the potential limiting mixed Hodge structure and a standard computation gives that

$$\mathrm{Ext}^1_{MHS}\left(H^1(-1), H^2\right)$$

contains a factor constructed from the subspace $Hg(Y,\mathbb{Z})$ in H^2. It then follows that the information contained in the level 1 extension data in the LMHS is essentially the

$$AJ_C(p_i - p_j) \in J(C)$$

Topology: The above examples of uses of Hodge theory were to geometric constructions arising from Linear algebra Hodge theoretic data. The original application of Hodge theory, meaning now the existence of fundamt functorial Hodge Structure on the cohomology of a smooth projective variety $X^n \subset \mathbb{P}^N$ were to topology letting $Y = \mathbb{P}^{N-1} \cap X$ be a general hyperplane section there is the;

(III.2) Lefschetz theorem ([L])

The induced mapping $H^q(X, \mathbb{Z}) \longrightarrow H^q(Y, \mathbb{Z})$ is an isomorphism for $q \leqq n-2$ and is injective for $q = n-1$.

This result was proved by Picard ([P]) when $n=2$ (algebriac surfaces) and Lefschetz's argument for the general case was an extension of that of Picard. It made use of Lefschetz pencils (Y_t) given by the $\mathbb{P}^1$ hyperplane sections containing a general $\mathbb{P}^{N-2}$. There are finitely many singular Y_{t_α}'s where the hyperplane is tangent to X, and the monodromy around these (Picard-Lefschetz transformations) plays a central role in the analysis. The result is a topological one; no use is made for Hodge theory. The other result of Lefschetz concerns the mapping

(III.3) $h^k : H^{n-k}(X, \mathbb{Q}) \longrightarrow H^{n+k}(X, \mathbb{Q})$

where $h \in H^2(X, \mathbb{Q})$ is dual to the homology class defined by Y.

III.4 Hard Lefschetz theorem ([L] and [H]):

The mapping h^k in (III.3) is an isomorphism. The reason for the "hard" is that in many senses this result is deeper than (III.2). In fact, Lefschetz's original topological argument was incomplete. It was partly in seeking to give a proof of (III.4) that Hodge [H] originated Hodge theory; the result itself is a Hodge-theoretic, not a topological one. Interestingly, Picard stated and gave a correct proof of (III.4) when $n=2$. His argument used the Poincaré complete reductibilty theorem ([P]) for abelian varieties, applied here to the family of Jacobian varieties $J(Y_t)$. The complete reductibility theorem is basically a Hodge-theoreic result and its higher dimensional analogue had to await the construction of PHS's on the $H^{n-1}(Y_t)$ for $n \geqslant 3$.

Vanishing Theorems

The Italian algebriac geometers had extraordinary geometric intuition and deep knowledge of examples. In its more mature period there were things they intuitively suspected but were unable to fully prove, and even some mistakes appeared. It has been said that what they were missing was H^1. The meaning of this is illustrated by the

Example: Let $L \longrightarrow X$ be a line bundle over a smooth algebriac surface and $C \subset X$ a curve. Various forms of the issue of showing that all of $H^0(C,L)$ comes by restriction from $H^0(X,L)$ kept arising. The Italians knew that this is not true in general. With the advent of sheaf theory and the long exact chomology sequence we now know this will be ok if one has the vanishing theorem.

$$H^1(X, L(-C)) = 0$$

* This issue does not arise as much for points on a curve because in this case the dual of H^1 is an H^0

Thus one needs the formalization of sheaf theory chomology and vanishing theorems. The thesis of this last part of the talk is

The Lefschetz theorem (III.2) plus the existence of Hodge Structure on ordinary chomology imply vanishing theorems.
More precisely, one needs the existence of the Hodge structure.

$$(III.5) \quad H^r(X, \mathbb{C}) \cong \bigoplus^{p+q=r} H^{p,q}(X)$$

plus the isomorphism

$$(III.6) \quad H^{p,q}(X) \cong H^q(\Omega^p_X)$$

This principle is well known since the 1950's and over the years has attained a much extended and more widely applicable version. Here we will simply illustrate it in a very special case.

APPENDIX: A Torelli theorem

→ An I- surface is a smooth general type surface X that satisfies $K_X^2 = 1$, $q(X) = 0$, and $p_g(X) = 2$. These surfaces are well known classically. We shall derive the theorem.

→(i)← The local Torelli property is valid for any I- surface

→(ii)← The period mapping is $\phi : M_I \longrightarrow \Gamma | D_I$ where

$$\dim M_I = \left(\frac{1}{2}\right)\left(\dim D_I - 1\right) = 28,$$

and where $\phi(M_I)$ is a maximal integral manifold of the IPR on D_I, which is contact system.